# Exact solutions for nondiffracting beams. I. The scalar theory

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Received June 12, 1986; accepted November 24, 1986

We present exact, nonsingular solutions of the scalar-wave equation for beams that are nondiffracting. This means that the intensity pattern in a transverse plane is unaltered by propagating in free space. These beams can have extremely narrow intensity profiles with effective widths as small as several wavelengths and yet possess an infinite depth of field. We further show (by using numerical simulations based on scalar diffraction theory) that physically realizable finite-aperture approximations to the exact solutions can also possess an extremely large depth of field.

Any field of wavelength $\lambda$ initially confined to a finite area of radius $r$ in a transverse plane will be subject to diffractive spreading as it propagates outward from that plane in free space. The characteristic distance beyond which diffractive spreading becomes increasingly noticeable is $r^2/\lambda$, the Rayleigh range. For this reason it is commonly thought that any beamlike field (i.e., one whose intensity is maximal along the axis of propagation and that tends to zero with an increasing transverse coordinate) must eventually undergo diffractive spreading as it propagates. This is certainly true, for example, of Gaussian beams: a Gaussian beam having a spot size $r$ diverges at an angle proportional to $\lambda/r$ at distances $z \gg r^2/\lambda$ from the beam waist.[1]

We present here free-space, beamlike, exact solutions of the wave equation that are not subject to transverse spreading (diffraction) after the plane where the beam is formed. These solutions are nonsingular and, like plane waves, have finite energy density rather than finite energy. Most importantly, they can have sharply defined intensity distributions as small as several wavelengths in every transverse plane, independent of propagation distance.

We begin with the wave equation for free space:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)E(\mathbf{r}, t) = 0. \quad (1)$$

One can easily verify that an exact solution of Eq. (1) for scalar fields propagating into the source-free region $z \geq 0$ is

$$E(x, y, z \geq 0, t) = \exp[i(\beta z - \omega t)]\int_0^{2\pi} A(\phi)\exp[i\alpha(x\cos\phi + y\sin\phi)]\mathrm{d}\phi, \quad (2)$$

where $\beta^2 + \alpha^2 = (\omega/c)^2$ and $A(\phi)$ is an arbitrary complex function of $\phi$. When $\beta$ is real, Eq. (2) represents a class of fields that are nondiffracting in the sense that the time-averaged intensity profile at $z = 0$,

$$I(x, y, z \geq 0) = \tfrac{1}{2}|E(\mathbf{r}, t)|^2 = I(x, y, z = 0), \quad (3)$$

is exactly reproduced for all $z > 0$ in every plane normal to the $z$ axis.

The only nondiffracting field [Eq. (2)] having axial symmetry is that for which $A(\phi)$ is independent of $\phi$, namely, a field whose amplitude is proportional to

$$E(\mathbf{r}, t) = \exp[i(\beta z - \omega t)]\int_0^{2\pi}\exp[i\alpha(x\cos\phi + y\sin\phi)]\frac{\mathrm{d}\phi}{2\pi} = \exp[i(\beta z - \omega t)]J_0(\alpha\rho). \quad (4)$$

Here $\rho^2 = x^2 + y^2$ and $J_0$ is the zero-order Bessel function of the first kind. When $\alpha = 0$ the solution is simply a plane wave, but for $0 < \alpha \leq \omega/c$ the solution is a nondiffracting beam whose intensity profile decays at a rate inversely proportional to $\alpha\rho$, as shown in Fig. 1. The effective width of the beam is determined by $\alpha$, and when $\alpha = \omega/c = 2\pi/\lambda$ (the maximum possible value for a nonevanescent field) the central spot assumes its minimum possible diameter of approximately $3\lambda/4$.

Since the intensity distribution of a $J_0$ beam decays as $1/\rho$, it is not square integrable. In fact, even though the intensity profile is sharply peaked, the amount of energy in each ring (i.e., between two consecutive zeros of the Bessel function) is approximately equal to that contained in the central maximum. It would therefore require an infinite amount of energy to create a $J_0$ beam over an entire plane. One can, however, create such a beam over a finite area, and we will now examine the propagation properties of $J_0$ beams of finite aperture by using scalar diffraction theory.

It is well known[2] that scalar diffraction theory yields excellent results when the wavelength is small compared with the size of the aperture and the propagation angles are not too steep. Both of these criteria are well satisfied in the following cases, and we have used the Rayleigh–Sommerfeld Green's function to perform the numerical simulations of field propagation.

Let us assume that in the $z = 0$ plane we have a $J_0$ beam with a central spot diameter of 200 $\mu$m ($\alpha = 240.5$ cm$^{-1}$) and a total aperture radius of 2 mm, as shown in Fig. 2(a). Also shown in Fig. 2(a) is a Gaussian with a full width at half-maximum (FWHM) of 100 $\mu$m. The total energy in the $J_0$ beam is 10 times greater than that of the Gaussian beam. Figure 3 shows a numerical simulation of the propagation of the central peak intensity (i.e., the intensity at $\rho = 0$) for

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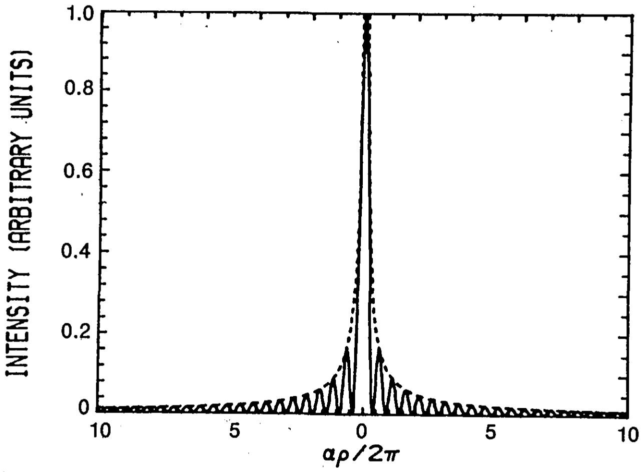
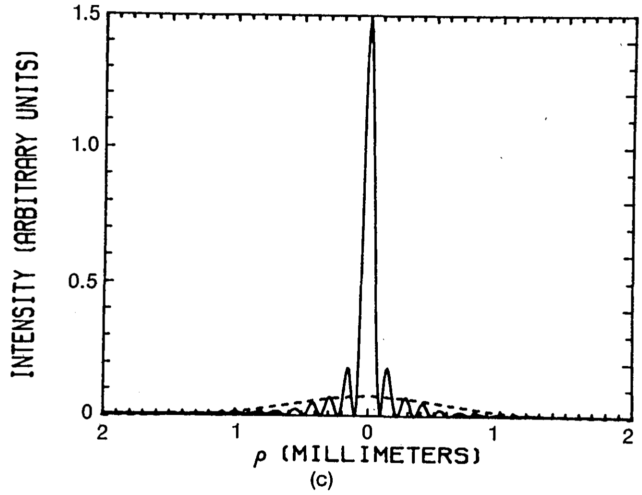



Fig. 1. Intensity distribution $J_0^2(\alpha\rho)$ (—) and its envelope function $2/\pi\alpha\rho$ (- - - -).



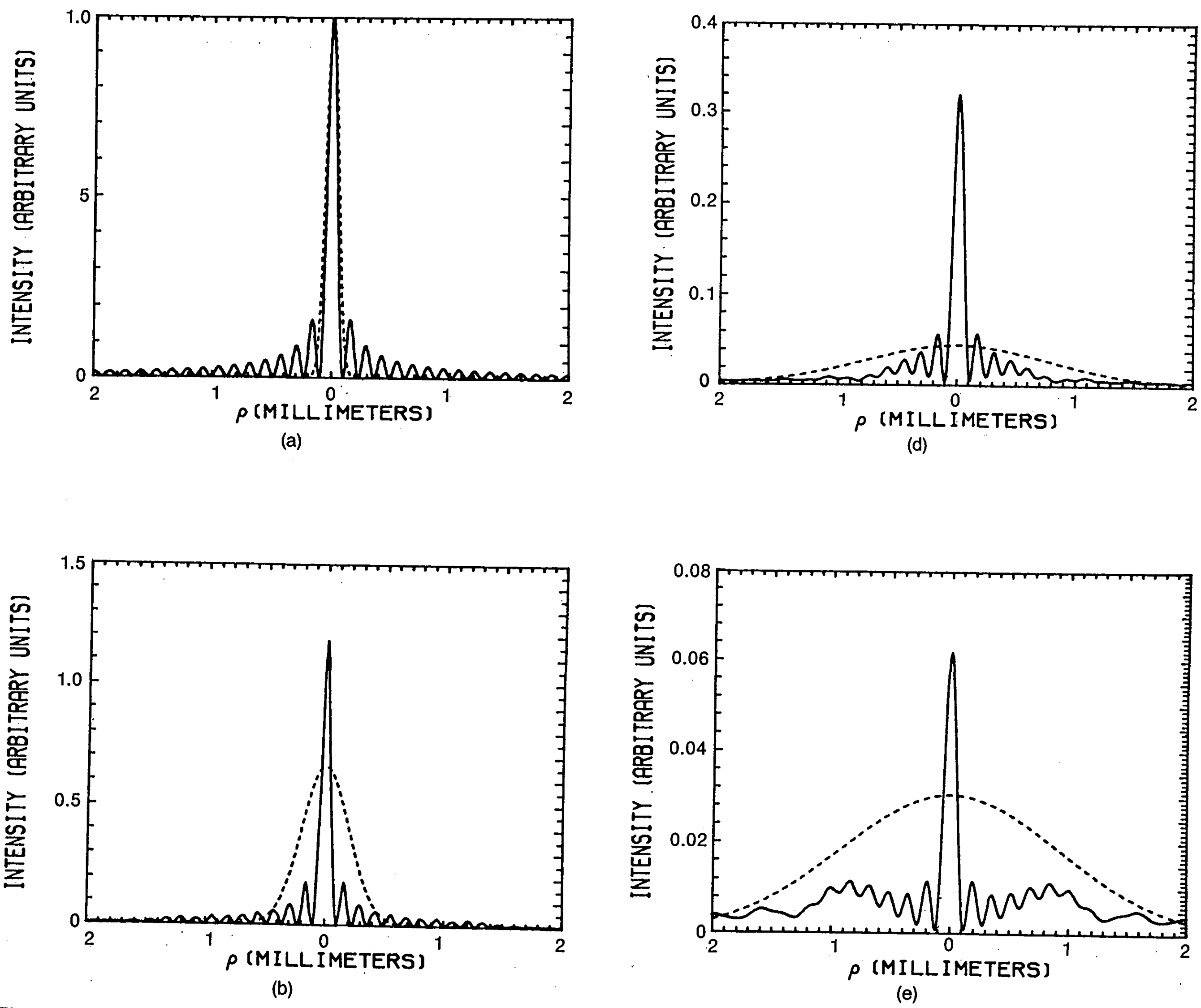




Fig. 2. Intensity distributions for a $J_0$ beam (—) and a Gaussian beam (- - - -): (a) when $z = 0$ (i.e., in the initial plane where the beams are assumed to be formed), (b) when $z = 25$ cm, (c) when $z = 75$ cm, (d) when $z = 100$ cm, and (e) when $z = 120$ cm, assuming that $\lambda = 0.5\ \mu$m. Note that in 2(b)–2(e) the intensity of the Gaussian beam has been multiplied by 10.

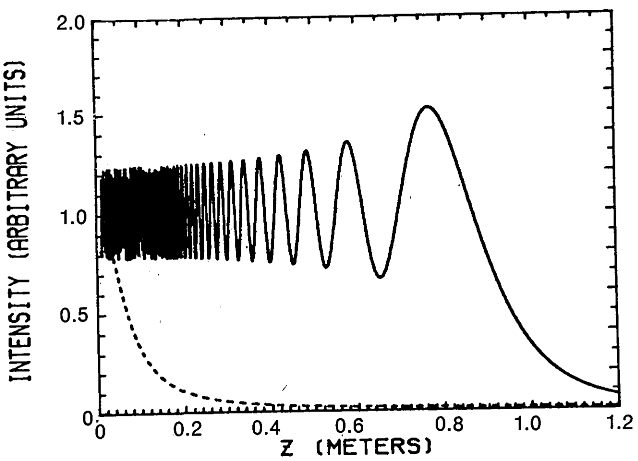



Fig. 3. Intensities $I(\rho = 0, z)$ at beam center, as a function of distance, of the $J_0$ (—) and Gaussian (- - - -) beams whose initial intensity distributions at $z = 0$ are shown in Fig. 2(a).

each beam as a function of distance from the $z = 0$ plane when $\lambda = 0.5\ \mu$m. The peak intensity of the $J_0$ beam oscillates in a manner reminiscent of the intensity distribution for the Fresnel diffraction pattern of a knife edge.

Figures 2(b), 2(c), 2(d), and 2(e) show the beam intensity profiles at $z = 25, 75, 100$, and 120 cm, respectively, with the intensity of the Gaussian profiles multiplied by a factor of 10 (that is, these Gaussian beam profiles are the result of a Gaussian in the $z = 0$ plane having a FWHM = 100 $\mu$m and a total energy equal to that of the $J_0$ beam). The $J_0$ beam has a remarkably greater depth of field than the Gaussian, and this is due in large part to its energy distribution. Only 5% of the total energy of the $J_0$ beam is initially contained within the central maximum, yet this is sufficient to create a sharply defined central spot with an unchanging 200-$\mu$m diameter over a distance of approximately 1 m. The Gaussian beam, on the other hand, initially concentrates almost 100% of its energy within the 200-$\mu$m spot diameter and shows measurable spreading after propagating only 1 cm.

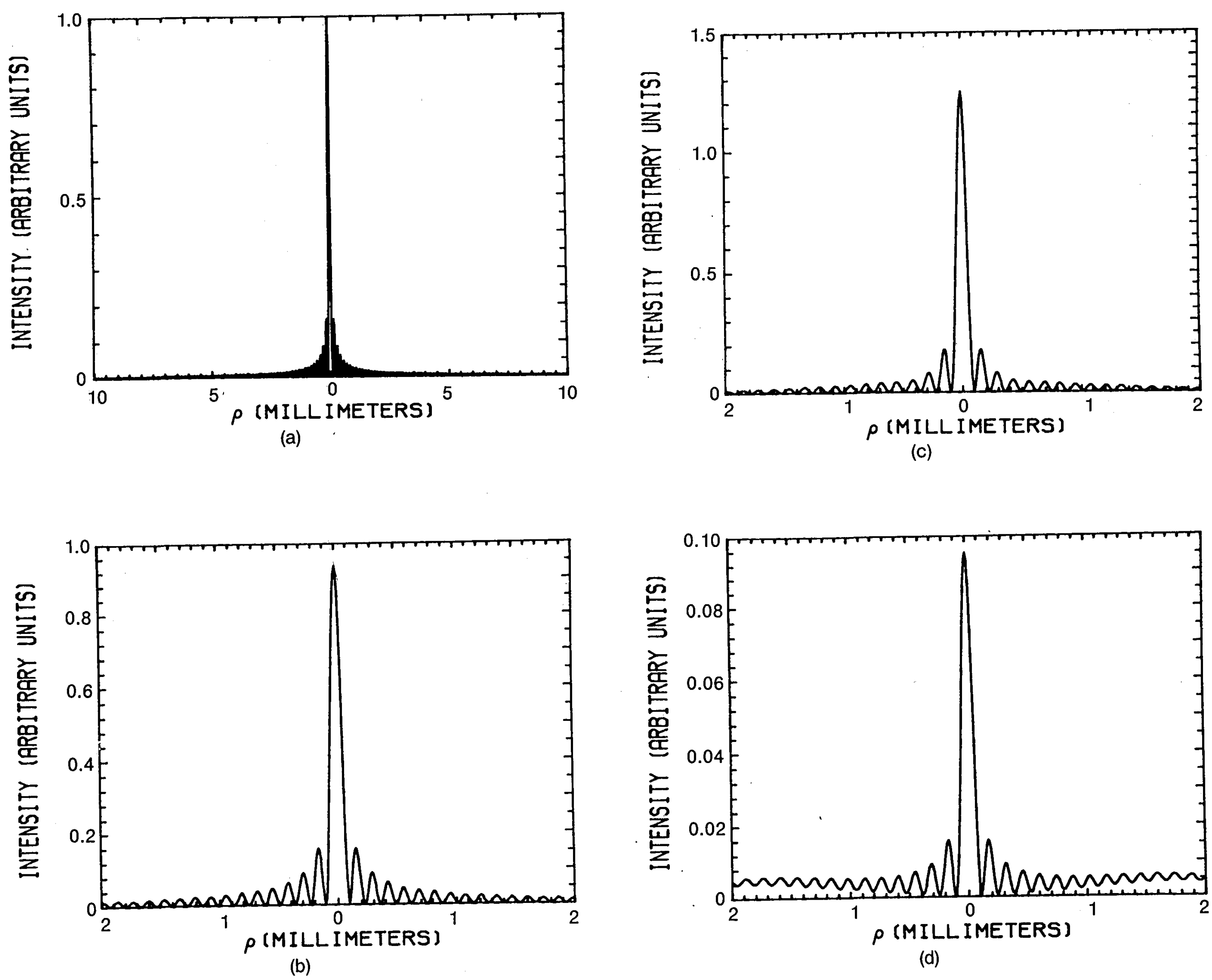



Fig. 4. Intensity distributions for a $J_0$ beam: (a) when $z = 0$, (b) when $z = 2$ m, (c) when $z = 4$ m, and (d) when $z = 5.5$ m. The aperture at $z = 0$ has a radius of 1 cm, but only the central 4 mm of the beam is plotted in (b)–(d) in order to show clearly that the central spot diameter has not changed.

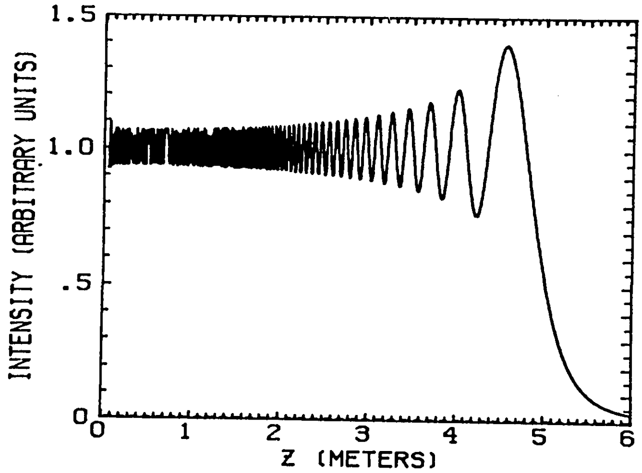



Fig. 5. Propagation of the central peak intensity $I(\rho = 0, z)$ of the $J_0$ beam shown in Fig. 4(a).

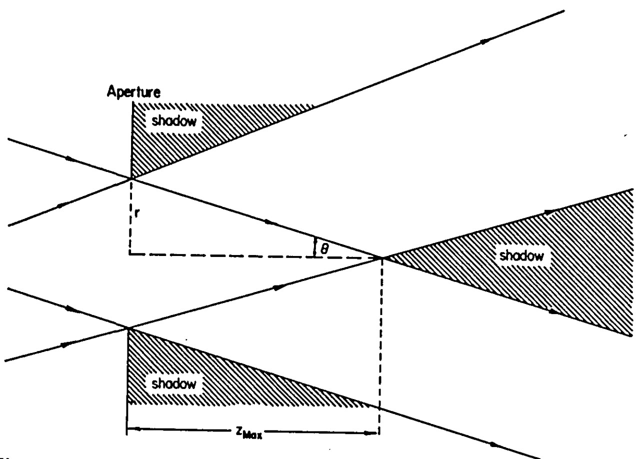



Fig. 6. Geometrical shadow zone for $J_0$ beams of finite aperture. A conical shadow zone begins at the distance $z = r/\tan\theta$, where $r$ is the radius of the limiting aperture at $z = 0$, $\theta = \sin^{-1}(\alpha\lambda/2\pi)$, and the diameter of the central maximum of the $J_0$ beam is approximately $3\pi/2\alpha$.

Let us now keep the central spot diameter of the $J_0$ beam fixed at 200 $\mu$m and increase the initial aperture radius from 2 mm to 1 cm, as shown in Fig. 4(a). That 200-$\mu$m spot will remain clearly visible and propagate more than 5 m without spreading, as demonstrated by the numerical simulations shown in Figs. 4(b)–4(d). Figure 5 shows that increasing the aperture by a factor of 5 not only increases the propagation range by that same amount but also decreases the magnitude of the fluctuation in peak intensity. A beam with such a great depth of field would be very useful, for example, in performing high-precision autocollimation or alignment.

There is a simple yet accurate method for finding the range of a $J_0$ beam of finite aperture. One sees from Eq. (4) that the $J_0$ beam is a superposition of plane waves, all having the same amplitude and traveling at the same angle $\theta = \sin^{-1}(\alpha\lambda/2\pi)$ relative to the $z$ axis but having different azimuthal angles ranging from 0 to $2\pi$ rad. For such a field geometrical optics predicts, as shown in Fig. 6, that a conical shadow zone begins at the distance

$$z_{\max} = r/\tan\theta = r[(2\pi/\alpha\lambda)^2 - 1]^{1/2}, \quad (5)$$

where $r$ is the radius of the aperture in which the $J_0$ beam is formed. For example, in the case we studied in Figs. 2 and 3, $\tan\theta = 1.9 \times 10^{-3}$, the initial aperture radius $r = 0.2$ mm, and we find that $z_{\max} = 1.05$ m. In the case studied in Figs. 4 and 5, the propagation angle $\theta$ remains the same, but $r$ is increased to 1 cm, and thus $z_{\max} = 5.25$ m. In both instances $z_{\max}$ corresponds to a point located at the base of the sharp final decay in the peak intensity of the $J_0$ beam.

In fact, Eq. (5) has been found to predict accurately the effective range of $J_0$ beams of finite aperture for all values of $\alpha$ in the range $\omega/c \geq \alpha \geq 2\pi/r$. When $\alpha > \omega/c$, the wave is evanescent, and $z_{\max} = 0$. When $\alpha < 2\pi/r$, the source field is essentially just a disk of radius $r$, and $z_{\max}$ equals the Rayleigh range.

In part I of this investigation we have presented only the scalar theory of nondiffracting beams; in part II we will present the complete electromagnetic field theory. Several methods of creating a $J_0$ beam of finite aperture appear to be feasible, and preliminary experimental confirmations of the predictions presented here have been obtained. Further experiments are under way, and detailed results will be presented separately.[3]

Finally, we want to point out that the class of nondiffracting fields given by Eq. (2) can be further generalized to include polychromatic solutions. One can easily show that any linear superposition of nondiffracting fields [Eq. (2)], all having the same $\alpha$ but having different frequencies $\omega \geq c\alpha$, is still nondiffracting in the sense that the time-averaged intensity distribution is the same in every plane normal to the $z$ axis. The types of transversely nondiffracting pulses that can be constructed are currently being studied.

## ACKNOWLEDGMENTS

I thank J. H. Eberly and E. Wolf for some very helpful discussions.

## REFERENCES

1. See, for example, H. Kogelnik and T. Li, "Laser beams and resonators," Proc. IEEE, **54**, 1312–1392 (1966).
2. See, for example, C. J. Bouwkamp, "Diffraction theory," Rep. Prog. Phys., **17**, 35–100 (1954).
3. A brief report on some of our initial experimental results was presented at the 1986 Annual Meeting of the Optical Society of America: J. Durnin, J. J. Miceli, and J. H. Eberly, "Diffraction-free beams," J. Opt. Soc. Am. A **3**, P128 (1986).