Connection Between X Waves, Fourier-Bessel Series And Optimal Modelling Aperture For Circular Symmetric Arrays

Paul D Fox
INSERM U619 & CIT
Hôpital Bretonneau CHRU Tours
2 Bis Boulevard Tonnellé
37044, Tours, France.
fox@med.univ-tours.fr

Jian-yu Lu
Department of Bioengineering
University of Toledo
2801 West Bancroft Street
Toledo, Ohio 43606-3390, USA.
jilu@eng.utoledo.edu

Sverre Holm
Department of Informatics
University of Oslo
P.O. Box 1080, Blindern
N-0316 Oslo, Norway.
sverre@ifi.uio.no

François Tranquart
INSERM U619 & CIT
Hôpital Bretonneau CHRU Tours
2 Bis Boulevard Tonnellé
37044, Tours, France.
tranquart@med.univ-tours.fr

***Abstract*— This study concerns two issues. Firstly an equivalence is given between the use of 1D Fourier-Bessel series and the X Wave Transform, as applied to annular arrays and the computation and tuning of linear fields. Secondly, an optimal choice of modelling aperture in the Fourier-Bessel analysis is suggested. The Fourier-Bessel approach is a numerical method which has been developed and implemented for calculation and tuning of linear lossless fields in both continuous wave and pulsed wave cases. The X Wave Transform is a theoretical mathematical method for expressing an arbitrary linear lossless fields as a summation of weighted X Wave expansions. Our discussion establishes a connection between these two studies, showing that the numerical Fourier-Bessel approach is in the limit equivalent to the analytical X wave approach to the description of circular symmetric fields. Furthermore it suggests a numerical optimisation of the Fourier-Bessel approach by implementing of a one-off optimal modelling aperture which replaces the previous iterative approach. Potential applications exist in the area of optimal ultrasound field calculation and tuning, for example in the study of contrast agent responses.**

I. Introduction

We study here the connection between the use [1], [2] of one dimensional Fourier-Bessel Series (FBS) [3], [4] and the X Wave Transform (XWT) [5] with respect to circular symmetric fields. In both cases the respective techniques allow the propagated field to be described as a set of multifrequency nondiffracting J_0 Bessel beams [6], [7], with the Fourier-Bessel approach representing a numerical implementation of the theoretical X wave approach. Theoretically, nondiffracting beams such as Bessel beams and X waves can propagate to an infinite distance without spreading if they are produced with an infinite aperture and energy. In practice, when the aperture and energy are always finite, they still have a large depth of field. Here we show that the FBS approach is an equivalent numerical implementation to the XWT approach, and a subsequent numerical optimisation of the FBS approach is then also suggested. The practical relevance of the study resides in the area of field calculation and tuning methods.

II. X Wave Theory

The circular symmetric wave equation in cylindrical coordinates for source-free, lossless, and isotropic-homogeneous media is given by

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\Phi(r,z,t)=0 \qquad (1)$$

where $\Phi(r,z,t)$ denotes acoustic pressure at a spatial point (r,z) and time t, z is the axial axis, and c is the speed of sound in the medium. A solution to (1) is an 0^{th} order X wave [8] :

$$X_\zeta(r,z,t)=\int_0^\infty T_{0,\zeta}(k)B_{k,\zeta}(r,z,t)dk$$
$$B_{k,\zeta}(r,z,t)=J_0(kr\sin\zeta)e^{jk\cos\zeta(z-c_1t)} \qquad (2)$$

where $T_{0,\zeta}(k)$ is the weighting function and $B_{k,\zeta}(r,z,t)$ is a 0^{th} order Bessel beam solution to (1). The axial propagation velocity $c_1=c/\cos\zeta$ is both the group and phase velocity of the X wave ($d\omega/dk_z=\omega/k_z=c_1\geq c$, where $k_z=k\cos\zeta$), $k=\omega/c$ is the wavenumber, $\omega=2\pi f$ is the angular frequency, f is the temporal frequency, $0\leq\zeta<\pi/2$ is the Axicon angle [9]–[11] of the X wave, $J_0(\cdot)$ is the 0^{th} order Bessel function of the first kind, and $T_{0,\zeta}(k)$ is a weighting function which may be related to the transfer function of a practical acoustic transducer for a given ζ.

Lu and Liu [5] considered the generalised solution $\Phi(r,z,t)$ to equation (1) for all nonevanescent waves as the summation of all possible 0^{th} order X waves $X_\zeta(r,z,t)$ in (2) integrated over the free parameter ζ :

$$\Phi(r,z,t)=\int_0^{\pi/2}\int_0^\infty T_{0,\zeta}(k)B_{k,\zeta}(r,z,t)dkd\zeta \qquad (3)$$

and showed that any arbitrary (well behaved physically realisable) field $\Phi(r,z,t)$ could be represented in the form of (3) by appropriate selection of the weighting function $T_{0,\zeta}(k)$. For a given field this is found by inverting (3) to obtain

$$T_{0,\zeta}(k) = \frac{k^2 c \sin\zeta \cos\zeta H(k)}{2\pi} \times \int_{-\infty}^{\infty}\int_{0}^{\infty} \Phi(r,z,t) B^*_{k,\zeta}(r,z,t) r dr dt \tag{4}$$

where $B^*_{k,\zeta}(r,z,t)$ is the complex conjugate of $B_{k,\zeta}(r,z,t)$, $H(k)$ represents the Heaviside step function and $T_{0,\zeta}(k)$ is a special case of (26) in [5]. Equations (3) and (4) together are the X Wave Transform, and we now aim to derive the equivalence between it and the recent use of 1D Fourier-Bessel series [3], [4] for circular symmetric fields [1], [2], [12]–[14].

III. Fourier-Bessel Theory

A 1D Fourier-Bessel series [3], [4] for the variable r may be used to model a transducer pressure function of the type $q(r,z_0,\omega)$, where $q(r,z_0,\omega)$ denotes the radial component of the field variation at an angular frequency ω. Using the series, $q(r,z_0,\omega)$ may be represented by

$$q(r,z_0,\omega) = \sum_{i=1}^{\infty} A_i(\omega,a) J_0(\alpha_i r)$$
$$\alpha_i = x_i/a \quad : \quad J_0(x_i) = 0 \quad : \quad 0 \le r \le a \tag{5}$$

where $J_0(\cdot)$ is the Bessel function of the first kind of order zero. This series applies over the range $0 \le r \le a$ for any choice of modeling aperture a, subject to $q(r,z_0,\omega)$ at $r = a$ satisfying the necessary boundary condition $q(a,z_0,\omega) = 0$ arising from $J_0(\alpha_i a) = J_0(x_i) = 0$ for all i. (We use the term *aperture* here to refer to the modeling radius a rather the full diameter $2a$). For annular arrays we may select any value $a > R$ since the (relative) surface pressure $q(r,z_0,\omega)$ is taken as zero by definition beyond the outer edge of the transducer $r > R$ in the plane of the transducer at $z_0 = 0$. The roots x_i in (5) are the known infinite set of (real) monotonically increasing positive solutions to $J_0(x_i) = 0$, and the corresponding scaled Bessel basis parameters $\alpha_i = x_i/a$ cause the basis functions $J_0(\alpha_i r)$ in (5) to become orthogonal such that the weighting coefficients $A_i(\omega,a)$ are given by

$$A_i(\omega,a) = \frac{2}{a^2 J_1^2(x_i)} \int_0^a q(r,z_0,\omega) J_0(\alpha_i r) r dr \tag{6}$$

which, for the case of N-ring annular arrays with complex ring quantisation levels $q_p(\omega)$ for $p = 1,\ldots,N$ and inner and outer radii r_p^- and r_p^+ become

$$A_i(\omega,a) = \frac{2}{a x_i J_1^2(x_i)} \sum_{p=1}^{N} \left[r_p^+ J_1(\alpha_i r_p^+) - r_p^- J_1(\alpha_i r_p^-) \right] q_p(\omega) \tag{7}$$

Then (see [1], [2], [12]–[14]), if $q(r,z_0,\omega)$ is considered to be a given (source) field at z_0, then based on the form of (5) and (2) one may propose a limited diffraction field estimate $\hat{q}(r,z,\omega)$ of $q(r,z,\omega)$ at distance z for frequency ω as

$$\hat{q}(r,z,\omega) = \sum_{i=1}^{\infty} A_i(\omega,a) J_0(\alpha_i r) e^{jz\sqrt{k^2-\alpha_i^2}} \tag{8}$$

When $\alpha_i \le k$, then $\sqrt{k^2-\alpha_i^2}$ is real and hence $e^{jz\sqrt{k^2-\alpha_i^2}}$ represents an oscillatory propagation to infinity in the z direction since z is real. However, for the case of $\alpha_i > k$, then $\sqrt{k^2-\alpha_i^2}$ becomes imaginary and $e^{jz\sqrt{k^2-\alpha_i^2}}$ corresponds generally to evanescent waves. Therefore, since the scaling parameters $\alpha_i = x_i/a$ in (5) increase monotonically with index i for a given value of a, the evanescent feature means that in practice we may truncate the infinite sum in i to only the value $l(k,a)$ at which $\sqrt{k^2-\alpha_i^2}$ switches between being real and imaginary. This limit may always be found numerically, and was derived analytically in [2] as

$$l(k,a) \approx ka/\pi + 1/4 \tag{9}$$

Hence truncating the sum for i in (8) to the limit $l(k,a)$ gives

$$\hat{q}(r,z,\omega) = \sum_{i=1}^{l(k,a)} A_i(\omega,a) J_0(\alpha_i r) e^{jz\sqrt{k^2-\alpha_i^2}} \tag{10}$$

and the inverse Fourier transform the time domain estimate

$$\hat{q}(r,z,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{q}(r,z,\omega) e^{-j\omega t} d\omega \tag{11}$$

Notice here that since we have applied (6) over a generally finite modelling aperture a, (10) represents a weighted set of limited diffraction beams. However, if we allow $a \to \infty$ then each limited diffraction beam above becomes a nondiffracting solution to (1) and the proposition is then that the estimate $\hat{q}(r,z,\omega)$ in the limit as $a \to \infty$ becomes the true field $q(r,z,\omega)$, namely

$$q(r,z,\omega) = lim_{a\to\infty} \sum_{i=1}^{l(k,a)} A_i(\omega,a) J_0(\alpha_i r) e^{jz\sqrt{k^2-\alpha_i^2}} \tag{12}$$

with time domain $q(r,z,t)$ from the inverse Fourier transform

$$q(r,z,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} q(r,z,\omega) e^{-j\omega t} d\omega \tag{13}$$

IV. Equivalence between Fourier-Bessel Series and X Wave Transform

The aim now is to show that the estimate $\hat{q}(r,z,t)$ from (11) as implemented numerically via (10) with $a \to \infty$ does in fact equate to the X wave solution (3) to (1), since then we know that the exact field solution is being computed as $a \to \infty$. This will be illustrated by converting the discrete sum in (12) into a continuous integral as $a \to \infty$. Begin by substituting from (6) into (12) to obtain

$$q(r,z,\omega) = lim_{a\to\infty} \sum_{i=1}^{l(k,a)} \frac{2}{a^2 J_1^2(x_i)} \cdot I(r,z,\omega,a,i) \tag{14}$$

where

$$I(r,z,\omega,a,i) = e^{jz\sqrt{k^2-\alpha_i^2}} J_0(\alpha_i r) \int_0^a q(r,z_0,\omega) J_0(\alpha_i r) r dr \tag{15}$$

Then utilise the $J_1(x_i)$ property from [3], [4]

$$J_1(x_i) \rightarrow \sqrt{2/\pi x_i}\cos(x_i - 3\pi/4) \quad , \quad i \rightarrow \infty \tag{16}$$

to give the fraction in (14) as

$$\frac{2}{a^2 J_1^2(x_i)} \rightarrow \frac{x_i}{a} \cdot \frac{\pi}{a} \cdot \frac{1}{cos^2(x_i - 3\pi/4)} \tag{17}$$

The first term on the right hand side of above may be recognised directly as $\alpha_i = x_i/a$. The second and third terms then benefit from utilising

$$J_0(x_i) \rightarrow \sqrt{2/\pi x_i}\cos(x_i - \pi/4) \quad , \quad i \rightarrow \infty \tag{18}$$

such that the roots x_i of $J_0(x_i) = 0$ become the roots of the cosine function $\cos(x_i - \pi/4) = 0$, namely

$$x_i \rightarrow \pi i - \pi/4 \quad , \quad i \rightarrow \infty \tag{19}$$

Then consider $\Delta\alpha_i = \alpha_i - \alpha_{i-1} = (x_i - x_{i-1})/a$. From above $\Delta\alpha_i \rightarrow \pi/a$ as $i \rightarrow \infty$ and $x_i - x_{i-1} \rightarrow \pi$ such that $cos^2(x_i - 3\pi/4) \rightarrow 1$. Hence with $\alpha_i = x_i/a$, (17) becomes

$$\frac{2}{a^2 J_1^2(x_i)} \rightarrow \alpha_i \Delta\alpha_i \quad , \quad i \rightarrow \infty \tag{20}$$

such that

$$q(r,z,\omega) = lim_{a\rightarrow\infty} \sum_{i=1}^{l(k,a)} I(r,z,\omega,a,i)\alpha_i\Delta\alpha_i \tag{21}$$

and then from $\Delta\alpha_i \rightarrow \pi/a$ we have $\Delta\alpha_i \rightarrow d\alpha \rightarrow 0$ for $a \rightarrow \infty$ and the infinite discrete sum in (21) becomes the continuous integral

$$q(r,z,\omega) = \int_0^k I(r,z,\omega)\alpha d\alpha \tag{22}$$

where $I(r,z,\omega)$ is $I(r,z,\omega,a,i)$ in (15) with $a \rightarrow \infty$ and α_i replaced by α such that

$$I(r,z,\omega) = e^{jz\sqrt{k^2-\alpha^2}} J_0(\alpha r) \int_0^\infty q(r,z_0,\omega) J_0(\alpha r) r dr \tag{23}$$

and in which the integration from $\alpha = 0$ to $\alpha = k$ represents the limit of the summation from $i = 1$ to $i = l(k,a)$ corresponding to all nonevanescent waves. Substituting from (22) into (13) then gives

$$q(r,z,t) = \int_0^k \int_{-\infty}^\infty \frac{\alpha}{2\pi} I(r,z,\omega) e^{-j\omega t} d\omega d\alpha \tag{24}$$

Then express (24) in terms of Axicon angles ζ and wavenumbers $k = \omega/c$, rather than alpha values α and frequencies ω by substituting $\alpha = k\sin\zeta$, $d\alpha = k\cos\zeta d\zeta$, $d\omega = cdk$. Also, for systems with positive frequency content only, insert the Heaviside notation $\int_{-\infty}^\infty dk = \int_0^\infty H(k)dk$ may also be inserted also to obtain

$$q(r,z,t) = \int_0^{\pi/2} \int_{-\infty}^\infty \frac{k^2 c \sin\zeta\cos\zeta H(k)}{2\pi} I(r,z,\omega) e^{-jkct} dk d\zeta \tag{25}$$

Then substituting $I(r,z,\omega)$ with $\omega = kc$ and $\alpha = k\sin\zeta$ gives

$$\begin{aligned} q(r,z,t) = \int_0^{\pi/2} \int_0^\infty \frac{k^2 c \sin\zeta\cos\zeta H(k)}{2\pi} \times \\ \int_0^\infty q(r,z_0,\omega) J_0(kr\sin\zeta) r dr \times \\ J_0(kr\sin\zeta) e^{jk\cos\zeta(z-c_1 t)} dk d\zeta \end{aligned} \tag{26}$$

which has the same structure as (3) with $T_{0,\zeta}(k)$ given by

$$T_{0,\zeta}(k) = \frac{k^2 c \sin\zeta\cos\zeta H(k)}{2\pi} \int_0^\infty q(r,z_0,\omega) J_0(kr\sin\zeta) r dr \tag{27}$$

in which an exact equivalence with (4) can be obtained by substituting the inverse Fourier transform $q(r,z_0,\omega) = \int_{-\infty}^\infty q(r,z_0,t)e^{j\omega t}dt$ and the identity $e^{jzk\cos\zeta} . e^{-jzk\cos\zeta} = 1$ to give

$$\begin{aligned} T_{0,\zeta}(k) = \frac{k^2 c \sin\zeta\cos\zeta H(k)}{2\pi} \int_{-\infty}^\infty \int_0^\infty q(r,z_0,\omega) e^{jzk\cos\zeta} \times \\ J_0(kr\sin\zeta) e^{-jk\cos\zeta(z-c_1 t)} r dr dt \end{aligned} \tag{28}$$

namely

$$\begin{aligned} T_{0,\zeta}(k) = \frac{k^2 c \sin\zeta\cos\zeta H(k)}{2\pi} \times \\ \int_{-\infty}^\infty \int_0^\infty q(r,z,t) B_{k,\zeta}^*(r,z,t) r dr dt \end{aligned} \tag{29}$$

Note that this structure is precisely equivalent to the XWT definitions (3) and (4). Hence we obtain the central result that the Fourier-Bessel series is exactly equivalent to the X Wave Transform in the case of implementing the FBS modelling aperture $a \rightarrow \infty$.

V. Suggested Optimal Modelling Aperture

The implemented field estimate $\hat{q}(r,z,t)$ given by (10) and (11) requires a summation of terms from $i = 1$ to $i = l(k,a)$, where $l(k,a) \approx ka/\pi + 1/4$ as per (9). Thus, as $a \rightarrow \infty$, $l(k,a) \rightarrow \infty$ and an infinite amount of terms need to be evaluated. Clearly this is impractical from a computational point of view due to finite time constraints, and for this reason an iterative approach was taken in [13] and [14] to obtain final field and ring quantisation estimates as a function of steadily increasing a. This required a suboptimal technique of gradually increasing a from low levels upwards, meaning that repeated cycles of computations were needed in order to obtain convergence. In this section we suggest a geometrical argument for obtaining a one-off value of modelling aperture a which achieves the equivalent to convergence without needing to go through the iterative process.

We know that in the real world the transducer has radius R, and that the field along the plane of the transducer is zero, so there can be no field contributions from anywhere beyond the radius of the transducer. With the Fourier-Bessel series, the definition of the series assures that the transducer plane is correctly modelled in the range $0 < r \leq a$. However, it is not correctly modelled for $r > a$ and this means that in the Fourier-Bessel scenario, a nonzero pressure will in general be emitted for all radial distances $r > a$. The situation is shown schematically in Figure 1.

Consider a point $S(r_s, z_s)$ in space. Then in general, in the Fourier-Bessel setup, it will receive a contribution from all points $r > a$ as well as from any given point at some general radius r_p considered on the transducer surface. Specifically, we shall consider the outer radius r_p^+ of ring p (where $p = 1, \ldots, N$) since this represents the furthest distance from the transducer centre for a given ring surface pressure. Then, in order to calculate the field correctly at point $S(r_s, z_s)$ at some time t, it is necessary to only receive the contribution from the true physical source point at r_p^+ on the transducer surface, and not that contribution from the false Fourier-Bessel sources arising at $r > a$ in the source plane. Now, the contribution from the true source travels a distance d_p at a velocity c from source to $S(r_s, z_s)$, where $d_p^2 = (r_p^+ + r_s)^2 + z_s^2$. If we then count time from some common base time $t = 0$ and assume that the source point has a pulse of duration δ_p with some time delay τ_p before emission begins, then the time t_p at which the last contribution from the source arrives at $S(r_s, z_s)$ is given by $t_p = \tau_p + \delta_p + d_p/c$. However, at the same time there is a Fourier-Bessel component being emitted in general from time $t = 0$ from all radial positions $r > a$. Taking a false source at the point $r = a$ (or, in theory, an infinitely small distance beyond $r = a$), on the side of the field closest to $S(r_s, z_s)$, its contribution will travel a distance d_a where $d_a^2 = (a - r_s)^2 + z_s^2$ and will arrive at $S(r_s, z_s)$ at a time $t_a = d_a/c$. (Note that there will also be a contribution from all other points at $r = a$ around the transducer centreline, but these will all travel a further distance and hence take more time than the current point considered). Therefore, to ensure that the estimate at point $S(r_s, z_s)$ contains only true source contributions, we must choose a large enough to ensure that $t_p < t_a$. Substituting for the relevant values gives

$$a > r_s + \sqrt{\left[c(\tau_p + \delta_p) + \sqrt{(r_p^+ + r_s)^2 + z_s^2}\right]^2 - z_s^2} \quad (30)$$

Note that this condition must be evaluated for components of each ring $p = 1, \ldots, N$ since each ring has a different outer radius r_p^+, and in general also a different time delay τ_p and possible also different pulse duration δ_p. This means that a total of N different minimum values of a will be obtained from above, and from a practical point of view the maximum of all these values must be taken as the final optimum value of a in order to exclude all erroneous components from the entire set of rings present.

VI. Conclusions and Further Work

An equivalence between the numerically implementable FBS and the theoretical XWT as applied to circular symmetric lossless linear fields has been derived. FBS is equivalent to XWT when implemented over an infinite modelling aperture. Previously the FBS has been used to compute and tune linear fields by use of an iteratively increasing modelling aperture. Here, a suggestion for a more economical non-iterative modelling aperture has been provided. Potential application areas include optimal calculation and tuning of linear ultrasound fields, for which an extension to lossy media is also of interest.

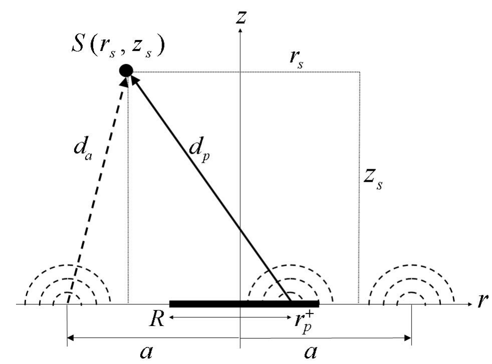

Fig. 1. Schematic diagram of point-to-point propagation under Fourier-Bessel modelling. Real source contribution from a given point r_p^+ on transducer surface and false source contributions from all points $r > a$

Acknowledgment

This work was sponsored principally by Le Studium, région Centre, France.

References

[1] S. Holm and P. D. Fox, "Analysis of Bessel beam quantisation in annular arrays," *Proc. 22nd Scandinavian Symposium on Physical Acoustics*, pp. 43–44, January 1999, Ustaoset, Norway, ISSN 1501-6773.
[2] P. D. Fox and S. Holm, "Decomposition of acoustic fields in quantised Bessel beams," *Ultrasonics*, vol. 38, pp. 190–194, March 2000.
[3] G. Tolstov, *Fourier Series*. New York: Dover Publications Inc., 1962.
[4] F. Bowman, *Introduction to Bessel Functions*. New York: Dover Publications Inc., 1958.
[5] J.-Y. Lu and A. Liu, "An X wave transform," *IEEE Trans. Ultrason., Ferroelect., Freq. Contr.*, vol. 47, no. 6, pp. 1472–1481, November 2000.
[6] J. A. Stratton, *Electromagnetic Theory*. New York and London: McGraw-Hill Book Company, 1941, p. 356.
[7] J. Durnin, "Exact solutions for nondiffracting beams. I. The scalar theory," *J. Opt. Soc. Am. A*, vol. 4, no. 4, pp. 651–654, 1987.
[8] J.-Y. Lu and J. F. Greenleaf, "Nondiffracting X waves - exact solutions to free-space scalar wave equation and their finite aperture realizations," *IEEE Trans. Ultrason., Ferroelect., Freq. Contr.*, vol. 39, no. 1, pp. 19–31, January 1992.
[9] J. H. Mcleod, "The Axicon: a new type of optical element," *J. Opt. Soc. Am.*, vol. 44, p. 592, 1954.
[10] C. B. Burckhardt, H. Hoffmann, and P. A. Grandchamp, "Ultrasound axicon: a device for focusing over a large depth," *J. Acoust. Soc. Am.*, vol. 54, no. 6, pp. 1628–1630, December 1973.
[11] M. S. Patterson and F. S. Foster, "Acoustic fields of conical radiators," *IEEE Trans. Sonics Ultrason.*, vol. SU-29, pp. 83–92, March 1982.
[12] P. D. Fox and S. Holm, "Modelling of cw annular arrays using limited diffraction Bessel beams," *IEEE Trans. Ultrason., Ferroelect., Freq. Contr.*, vol. 49, no. 1, pp. 85–93, January 2002.
[13] P. D. Fox, J. Cheng, and J.-Y. Lu, "Fourier-Bessel field calculation and tuning of a CW annular array," *IEEE Trans. Ultrason., Ferroelect., Freq. Contr.*, vol. 49, no. 9, pp. 1179–1190, September 2002.
[14] P. D. Fox, J. Cheng, and J.-Y. Lu, "Theory and experiment of Fourier-Bessel field calculation and tuning of a PW annular array," *J. Acoust. Soc. Am.*, vol. 113, no. 5, pp. 2412–2423, May 2003.