Construction of Limited Diffraction Beams with Bessel Bases

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Abstract — Limited diffraction beams have a large depth of field. They could have applications in medical imaging, tissue characterization, Doppler velocity estimation, nondestructive evaluation (NDE) of materials, as well as other physics related areas such as electromagnetics and optics. In this paper, a method is developed that uses limited diffraction beams discovered previously, such as, Bessel beams and X waves, as basis functions to construct new limited diffraction beams that may have practical usefulness. The method is implemented with linear matrix operations. Results show that the method is powerful and can obtain limited diffraction beams that are not intuitive to get directly from solving the wave equation.

I. INTRODUCTION

Since the discovery of localized waves [1] in 1983 and limited diffraction beams [2] in 1987, efforts have been made to develop new beams that have practical applications [3–19]. Because the previous studies are based on directly solving the isotropic-homogeneous wave equation, they are limited to a few simplified cases [5,19]. To obtain limited diffraction beams of desired properties, a general and practical method needs to be developed.

In this paper, a method that can obtain new limited diffraction beams of desired shapes is developed. This method uses the Bessel beams and X waves studied previously as basis functions to construct new beams. Matrix operations are used to determine the coefficients of the basis functions.

In next section, the method for construction of limited diffraction beams will be developed. Matrix implementation of the method will be given in section III. some preliminary results are shown in section IV. The conclusion is given in the final section.

II. METHOD

Both Bessel beams [2] and X waves [5,6] can be used as basis functions to construct new limited diffraction beams. Because X waves are special linear superpositions of Bessel beams in terms of frequency [5], for simplicity, only the Bessel basis functions will be used in this paper.

Theoretically, production of limited diffraction beams requires an infinite aperture. In practice, these beams can be well approximated with a finite aperture over a very large depth of field. If the aperture (diameter) of interest is D, the depth of field of both X waves and Bessel beams can be calculated [5].

Let's construct limited diffraction beams with linear superpositions of the Bessel beams (or the Bessel basis functions) [2,5] of the same depth of field within the finite aperture, D. The coefficients of the Bessel basis functions for the construction can be determined by minimizing some "distance" between the constructed beams and the desired beams or functions within the aperture of interest. If the "distance" is the least-squares error between the desired and constructed beams, the above construction can be represented by the least-squares formula [20]:

$$\min_{r\leq D} \|\Phi_{DT}(r,\phi) - \Phi_T(r,\phi)\|_2, \qquad (1)$$

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where $\Phi_{DT}(r, \phi)$ is the function of a desired shape in a plane (transverse plane) perpendicular to the beam axis, the subscript "DT" means "desired and transverse", $\Phi_T(r, \phi)$ is the transverse part of a designed (constructed) limited diffraction beam, (r, ϕ) are polar coordinates, and $\|\cdot\|_2$ represents L2 norm. With this method, one can construct new limited diffraction beams of both a desired depth of field and a desired transverse beam shape. Although in most cases the designed beams may not be exactly the same as desired beams, however, these beams are the "best" approximations to the desired beams in the sense of least-squares error within the aperture of interest.

With the Bessel basis functions [2,5], the designed limited diffraction beams in Eq. (1) can be written by

$$\begin{aligned}\Phi(r, \phi, z - c_1 t) &= \Phi_T(r, \phi) e^{i\beta(z - c_1 t)} \\ &= [D_0 J_0(\alpha r) + \sum_{n=1}^{N-1} D_n J_n(\alpha r) \cos n\phi \\ &+ \sum_{n=1}^{N-1} E_n J_n(\alpha r) \sin n\phi] e^{i\beta(z - c_1 t)}, \ (r \le D),\end{aligned} \quad (2)$$

where α is a scaling parameter (it is the same for all the basis functions so that the Bessel beams have the same depth of field), N is an integer, D_0, D_n, and E_n, $(n = 1, 2, \cdots, N-1)$, are coefficients to be determined, z is the axial distance, t is time, $\beta = \sqrt{(\omega/c)^2 - \alpha^2}$, ω is angular frequency, $c_1 = \omega/\beta$ is the phase velocity, and $J_n(\cdot)$ is the nth-order Bessel function of the first kind.

Because the transverse variables of the designed beams in Eq. (2) are separable from the axial variable and time, the desired beams can be chosen of the form:

$$\Phi_D(r, \phi, z - c_1 t) = \Phi_{DT}(r, \phi) e^{i\beta(z - c_1 t)}. \quad (3)$$

III. MATRIX IMPLEMENTATION

Substituting Eqs. (2) and (3) into Eq. (1), the least-squares formula can be represented by a linear system of equations [20]

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \quad (4)$$

where

$$\mathbf{A} = [a_{ij}], \quad (5)$$

$$\mathbf{x} = \begin{bmatrix} D_0 \\ D_1 \\ \cdots \\ D_{N-1} \\ E_1 \\ E_2 \\ \cdots \\ E_{N-1} \end{bmatrix}, \quad (6)$$

and

$$\mathbf{b} = \begin{bmatrix} \Phi_{DT}(r^0, \phi^0) \\ \Phi_{DT}(r^1, \phi^1) \\ \cdots \\ \Phi_{DT}(r^{M-1}, \phi^{M-1}) \end{bmatrix}, \quad (7)$$

and where M is the number of pixels of digitized transverse functions within the finite aperture, $\mathbf{A}$ is an $M \times (2N-1)$ matrix, $a_{ij} = J_j(\alpha r^i) \cos j\phi^i$ $\{(i = 0, 1, \cdots, M-1), (j = 0, 1, \cdots, N-1)\}$ and $a_{ij} = J_{j-N+1}(\alpha r^i) \sin (j - N + 1)\phi^i$ $\{(i = 0, 1, \cdots, M-1), (j = N, N+1, \cdots, 2N-2)\}$ are the elements of the matrix, $\mathbf{x}$ is a $(2N-1) \times 1$ vector variable, and $\mathbf{b}$ is an $M \times 1$ vector obtained from a desired transverse beam pattern.

From the normal equation,

$$\left(\mathbf{A}^T \mathbf{A}\right) \mathbf{x} = \mathbf{A}^T \mathbf{b}, \quad (8)$$

the coefficients, D_0, D_n, and E_n, $(n = 1, 2, \cdots, N-1)$, can be determined:

$$\mathbf{x} = \left(\mathbf{A}^T \mathbf{A}\right)^{-1} \mathbf{A}^T \mathbf{b}. \quad (9)$$

Eq. (9) is a least-squares solution to Eq. (1).

IV. RESULTS

With the above method, a question remains: how to choose the desired functions? The easiest way is to start with limited diffraction beams known previously and then modify them. This will be illustrated with the following examples.

Figs. 1 and 3 show the desired beams that are limited diffraction beams developed recently [7–9] and functions modified from them. The desired and the designed beams are marked as "desired" and "designed", respectively. Figs. 2 and 4 are line plots of the beams in Figs. 1 and 3, respectively, at some angles, ϕ, through the beam axis.

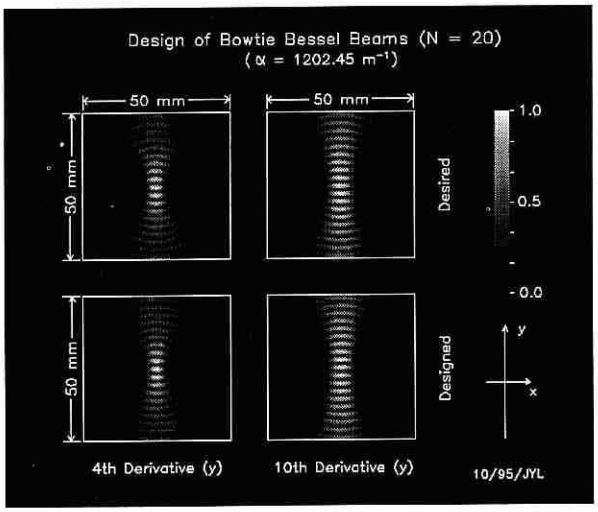

Fig. 1 Design of the 4th (panels in the left column) and the 10th (panels in the right column) derivative bowtie Bessel beams with Bessel basis functions. The desired and the designed beams are on the top and bottom, respectively. The absolute values of the beams are shown. The size of each panel is 50 mm $\times$ 50 mm, and the scaling factor, α, for both the bowtie Bessel beams and the Bessel basis functions is 1202.45 m^{-1}. The numbers of terms for the coefficients D_n and E_n are N and $N-1$, respectively, where $N = 20$. Notice that the beams are constructed only in the area of a diameter of 50 mm.

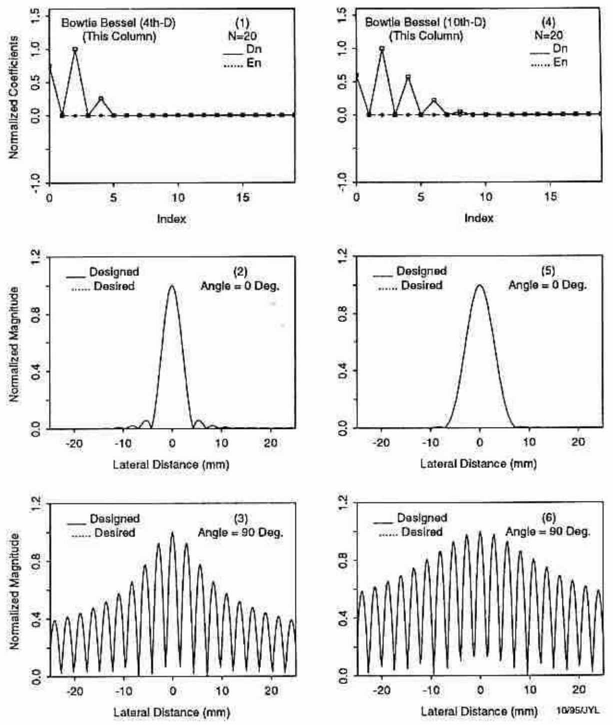

Fig. 2 Line plots of the 4th (panels in the left column) and the 10th (panels in the right column) derivative bowtie Bessel beams in Fig. 1 along the x ($\phi = 0°$) (panels in the middle row) and the y ($\phi = 90°$) (panels in the bottom row) axes. The designed and the desired beams are represented by full and dotted lines, respectively. These lines virtually overlap with each other although the number of terms of the Bessel basis functions used in the construction are small ($N = 20$). The coefficients, D_n (full lines with square points) and E_n (dotted lines with diamond points) are shown in the panels on the top row. Because D_n and E_n are about zero for $N \geq 9$, the number of terms of the Bessel basis functions can be smaller. The lateral axes of the panels in the bottom two rows are from -25 to 25 mm, and the vertical axes of these panels are normalized to 1.0. The lateral axes of the plots in the top row represent the index, n, and the vertical axes are normalized from -1 to 1.

The beams in Figs. 1 and 2 are called bowtie Bessel beams [7,8]. With these beams, bowtie X waves [7] can be constructed to dramatically reduce sidelobes of pulse-echo imaging systems while maintaining a very large depth of field. We have shown previously [7] that such imaging systems have sidelobes as low as those using conventional focused beams at their focuses but their depth of field are much larger. The layered array beams in Figs. 3 and 4 are newly developed [9]. These beams consist of well-structured layers and therefore have potential for real-time speed of sound imaging of scattering materials [9] (any point sound emitter traveling through the layered structures of the beams will modulate the signals of the receiver that has the layered beam response, and the period of the modulated signals can be used to determine the travelling speed of the emitter [10]).

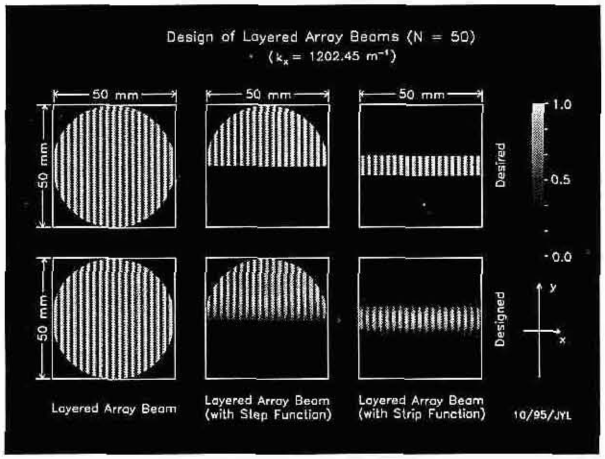

Fig. 3 Design of layered array beams (without modification (panels in the left column), modified with a step function (panels in the middle column), and modified with a strip function (panels in the right column)) with Bessel basis functions. The desired and the designed beams are on the top and bottom, respectively. The absolute values of the beams are shown. The size of each panel is 50 mm × 50 mm, and the scaling factor, k_x, of the layered array beams is equal to that of the Bessel basis functions, α (1202.45 m^{-1}). The number of terms of the Bessel basis functions is increased from that of Fig. 1 (N = 50).

It is noticed that when the desired beams are modified beams (the step and the strip layered array beams in the middle and right columns of Figs. 3 and 4), new limited diffraction beams that have similar shapes of the desired beams are constructed (see the bottom row of Fig.3). These new beams would be difficult to obtain directly from solving the wave equation.

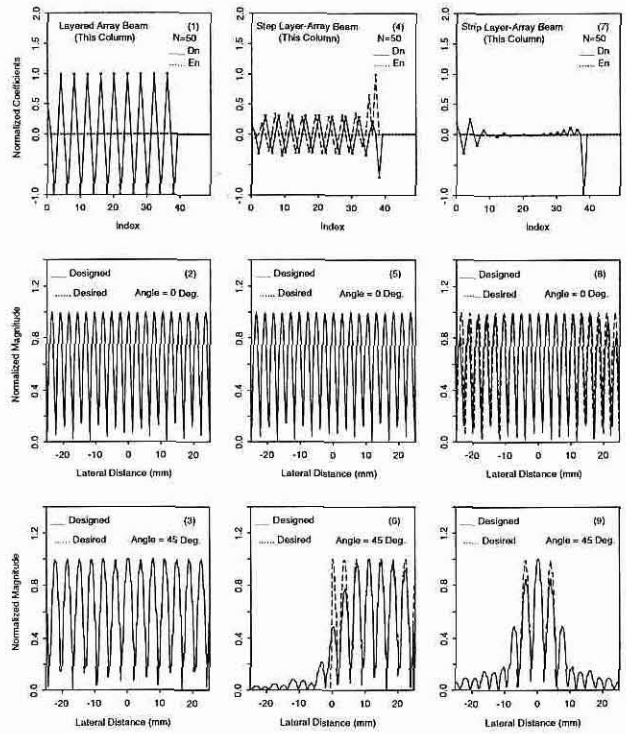

Fig. 4 Line plots of the layered array beams (without modification (panels in the left column), modified with a step function (panels in the middle column), and modified with a strip function (panels in the right column)) in Fig. 3 along the x ($\phi = 0°$) (panels in the middle row) and the diagonal ($\phi = 45°$) (panels in the bottom row) axes. This figure has the same format as Fig. 2 except that one more column is added. The non-smoothness of the line plots in the panels of the bottom row is caused by the nearest neighbor interpolations when plot along the diagonal axis.

V. CONCLUSION

We have developed a method to design limited diffraction beams that are the "best" approximations to desired functions in the sense of least-squares error within the aperture of interest. The preliminary results show that this method is robust and powerful in obtaining limited diffraction beams that have practical applications.

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VI. REFERENCES

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