Higher-order harmonics of limited diffraction Bessel beams

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We investigate theoretically the nonlinear propagation of the limited diffraction Bessel beam in nonlinear media, under the successive approximation of the KZK equation. The result shows that the nth-order harmonic of the Bessel beam, like its fundamental component, is radially limited diffracting, and that the main beamwidth of the nth-order harmonic is exactly $1/n$ times that of the fundamental. © *2000 Acoustical Society of America.* [S0001-4966(00)00503-8]

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INTRODUCTION

Bessel beams were first developed in 1941 by Stratton and were named undistorted progressive waves.[1] In the past decade, both Bessel beams and a more general type of beam called X wave have been widely investigated in the fields of acoustics and optics.[2,3] Theoretically, a J_0 Bessel beam (the lowest-order Bessel beam) with an infinite aperture has a beam profile of the *zero*th-order Bessel function of the first kind in the transverse plane and can travel to an infinite distance without changing its beam profile and amplitude. Numerical simulations and experiments show that even when produced with a finite aperture, this beam has a very large depth of field where the beam profile approximately maintains a J_0 Bessel function distribution. Because of these properties, the Bessel beam may have many potential applications,[2–6] such as ultrasonic imaging. It may also be applied to harmonic imaging developed recently.[3,5,7–9] In addition, the dispersion feature of the J_0 beam has been demonstrated to be applicable in nonlinear optics, where this beam can be viewed as a light beam with a tunable wavelength.[10]

In previous work, we studied theoretically the second harmonic generation of the Bessel beam.[7] Analysis indicates that the second harmonic of this beam is limited diffracting in the radial direction and the main beamwidth of the second harmonic is equal to one-half of that of the fundamental component in the Bessel field. In this paper, we investigate a more general case. It will be shown that for a J_0 Bessel beam, the harmonic of an order n is also radially limited diffracting and its main beamwidth is exactly $1/n$ times that of the fundamental component.

I. THEORY AND RESULTS

Assuming that an axial-symmetric source, with an angular frequency ω and a characteristic radius a, oscillates harmonically in time and that the sound absorption of the medium can be neglected (in attenuating medium, an exponential decay of the fundamental may occur), from the KZK (Khokhlov–Zabolotskaya–Kuznetsov) equation[7,11] and the perturbation method, we obtain the solution for the fundamental component of waves in terms of dimensionless variables[11]

$$\overline{q_l}(\xi,\eta)=\frac{2}{i\eta}\int_0^\infty \exp\left(i\,\frac{\xi^2+\xi'^2}{\eta}\right)J_0\left(\frac{2\xi\xi'}{\eta}\right)\overline{q_l}(\xi')\xi'\,d\xi', \tag{1}$$

and the solution for the nth-order harmonic component

$$\overline{q_n}(\xi,\eta)=\sum_{l=1}^{n-1}\overline{q}_{nlm}(\xi,\eta), \tag{2a}$$

where

$$\overline{q}_{nlm}(\xi,\eta)=\frac{n^2}{8}\int_{\eta'=0}^{\eta}\int_{\xi'=0}^{\infty}\frac{\xi'}{\eta-\eta'}\times\exp\left(\frac{in(\xi^2+\xi'^2)}{\eta-\eta'}\right)J_0\left(\frac{2n\xi\xi'}{\eta-\eta'}\right)\times\overline{q_l}(\xi',\eta')\overline{q_m}(\xi',\eta')d\xi'\,d\eta', \tag{2b}$$

and $n=l+m$. [Notice that in Eq. (2a) we have ignored the production of lower harmonics from higher harmonics because the pressure amplitude of the $(m+1)$th-order harmonic is assumed to be much smaller than that of the mth-order harmonic.] In these equations, $\xi=r/a$ and $\eta=2z/ka^2$ are the radial and axial dimensionless coordinates, $k=\omega/c$, and c is the speed of sound of medium. Correspondingly, the notations r and z denote the radial and axial coordinates. $\overline{q_l}(\xi')$ is the distribution function of the sound beam on the plane $\eta=0$. Equations (1) and (2) can be viewed as the complex-valued pressure amplitudes in a dimensionless form for the fundamental and nth-order harmonic components, respectively. These solutions are derived under the conditions that the Mach number $\varepsilon\ll1$ and $(ka)^2\gg1$.

Assume that the J_0 beam with a scaling parameter α has the form

$$\overline{q_l}(\xi')=J_0(\alpha\xi') \tag{3}$$

at the source. In previous work[7] it has been shown that the fundamental component of this beam is given by

$$\overline{q_l}(\xi,\eta)=J_0(\alpha\xi)\exp\left(-\frac{i}{4}\alpha^2\eta\right), \tag{4}$$

and the second harmonic under an asymptotic condition ($\alpha^2\eta$ is sufficiently large) can be expressed as

$$\overline{q_2}(\xi,\eta)=\sqrt{2}\left(\frac{e^{i3\pi/4}}{4\alpha\sqrt{\pi}}\right)J_0(2\alpha\xi)\,\eta^{1/2}\exp\left(-\frac{i\alpha^2}{2}\eta\right). \quad (5)$$

In the following, we will derive a more general case that the nth-order harmonic component of the Bessel beam has the $J_0(n\alpha\xi)$ function distribution in the radial distance. From Eqs. (4) and (5), we can assume in general that under the asymptotic condition ($\alpha^2\eta$ is sufficiently large) the lth- and mth-order harmonics of the Bessel beam are given by

$$\overline{q_l}(\xi,\eta)=A_lJ_0(l\alpha\xi)\,\eta^{(l-1)/2}\exp\left(-\frac{il}{4}\alpha^2\eta\right) \quad (6)$$

and

$$\overline{q_m}(\xi,\eta)=A_mJ_0(m\alpha\xi)\,\eta^{(m-1)/2}\exp\left(-\frac{im}{4}\alpha^2\eta\right), \quad (7)$$

respectively. From Eq. (2b), it follows that (notice that $\hat{q}_{mln}$ is different from $\overline{q_{mln}}$ by a constant)

$$\hat{q}_{nlm}(\xi,\eta)=\int_{\eta'=0}^{\eta}\int_{\xi'=0}^{\infty}\frac{\xi'}{\eta-\eta'}\times\exp\left(\frac{in(\xi^2+\xi'^2)}{\eta-\eta'}\right)J_0\left(\frac{2n\xi\xi'}{\eta-\eta'}\right)\times J_0(l\alpha\xi')J_0(m\alpha\xi')\times\exp\left(-\frac{in}{4}\alpha^2\eta'\right)\eta'^{(n/2)-1}\,d\xi'\,d\eta'. \quad (8)$$

To simplify, we change Eq. (8) first to a triple integral

$$\hat{q}_{nlm}(\xi,\eta)=\frac{1}{\pi}\int_{\eta'=0}^{\eta}\int_{\xi'=0}^{\infty}\int_{t'=0}^{\pi}\frac{\xi'}{\eta-\eta'}\times\exp\left(\frac{in(\xi^2+\xi'^2)}{\eta-\eta'}\right)J_0\left(\frac{2n\xi\xi'}{\eta-\eta'}\right)\times J_0(\lambda\alpha\xi')\exp\left(-\frac{in}{4}\alpha^2\eta'\right)\times\eta'^{(n/2)-1}\,d\xi'\,d\eta'\,dt', \quad (9)$$

where $\lambda=(l^2+m^2-2lm\cos t')^{1/2}$ and the formula

$$\pi J_0(X)J_0(x)=\int_0^{\pi}J_0[(X^2+x^2-2Xx\cos t)^{1/2}]dt \quad (10)$$

has been used. Applying the following

$$\int_0^{\infty}J_0(\alpha t)J_0(\beta t)e^{\pm i\gamma^2t^2}t\,dt=\pm\frac{i}{2}\gamma^{-2}\exp\left[\mp\frac{i}{4}\gamma^{-2}(\alpha^2+\beta^2)\right]J_0\left(\frac{1}{2}\alpha\beta\gamma^{-2}\right), \quad (11)$$

we have

$$\hat{q}_{nlm}(\xi,\eta)=\frac{i}{2\pi n}\int_{\eta'=0}^{\eta}\int_{t'=0}^{\pi}J_0(\lambda\alpha\xi)\exp\left(-\frac{i\alpha^2\lambda^2\eta}{4n}\right)\times\exp\left[-\frac{i\alpha^2}{4}\eta'\left(\frac{\lambda^2}{n}-n\right)\right]\eta'^{(n/2)-1}\,d\eta'\,dt'. \quad (12)$$

Formally, the integral about η' in the equation above can be expressed in terms of the incomplete Gamma function $P(a,z)$, which is defined by formula 6.5.1 from Ref. 12. We then obtain

$$\hat{q}_{nlm}(\xi,\eta)=\frac{2i}{\pi n}\exp\left(-\frac{in\alpha^2}{4}\eta\right)\int_{t=0}^{\pi/2}J_0(\lambda\alpha\xi)b^{-n/2}\times\exp(b\eta)\Gamma\left(\frac{n}{2}\right)P\left(\frac{n}{2},b\eta\right)dt, \quad (13)$$

with $b=i\alpha^2(lm/n)\cos^2 t$ and the transformed variable $t=t'/2$. In order to analyze Eq. (13), we concentrate on the function

$$f_n(t)=b^{-n/2}\exp(b\eta)\Gamma\left(\frac{n}{2}\right)P\left(\frac{n}{2},b\eta\right). \quad (14)$$

Note that this function resembles function (11) of Ref. 7, and its real and imaginary parts are extremely similar to the delta function under the asymptotic condition mentioned above. Therefore, Eq. (13) can be approximated well under this condition with

$$\hat{q}_{nlm}(\xi,\eta)=\frac{2e^{i3\pi/4}}{\sqrt{\pi}n^2\alpha}\exp\left(-\frac{in\alpha^2}{4}\eta\right)\times J_0(n\alpha\xi)\,\eta^{(n-1)/2}\sqrt{\frac{n}{lm}}. \quad (15)$$

From Eqs. (2) and (15), one obtains the nth-order harmonic component of the Bessel beam

$$\overline{q_n}(\xi,\eta)=A_nJ_0(n\alpha\xi)\,\eta^{(n-1)/2}\exp\left(-\frac{in}{4}\alpha^2\eta\right). \quad (16)$$

The coefficient A_n is given by the following recursive relationship [obtained by inserting both Eqs. (16) and (15) into Eq. (2a)]

$$A_n=\frac{\sqrt{n}e^{i3\pi/4}}{4\sqrt{\pi}\alpha}\sum_{l=1}^{n-1}A_lA_m\sqrt{\frac{1}{lm}}. \quad (17)$$

Let

$$A_n=\sqrt{n}\left(\frac{e^{i3\pi/4}}{4\sqrt{\pi}\alpha}\right)^{n-1}C_n, \quad (18)$$

from Eq. (17), we have

$$C_n=\sum_{l=1}^{n-1}C_lC_m \text{ or } C_n=\sum_{l=1}^{n-1}C_lC_{n-l}, \quad (19)$$

where C_n is the Catalan number. The first two terms of C_n are given by $C_1=1$ and $C_2=1$, and generally

$$C_n=\frac{1}{n}\binom{2n-2}{n-1}. \quad (20)$$

Finally, the nth-order harmonic component of the Bessel beam can be expressed by

$$\overline{q_n}(\xi,\eta)=\sqrt{n}C_n\left(\frac{e^{i3\pi/4}}{4\sqrt{\pi}\alpha}\right)^{n-1}\exp\left(-\frac{in\alpha^2}{4}\eta\right)\times J_0(n\alpha\xi)\,\eta^{(n-1)/2}. \tag{21}$$

Notice that Eq. (21) is obtained by assuming the interaction of nonlinear components in the nearfiled is negligible so that the asymptotic results in Eqs. (6) and (7) can be inserted into Eq. (2) to get Eq. (8).

II. DISCUSSION

From Eq. (21), we see that under the asymptotic condition ($\alpha^2\eta$ is sufficiently large) the nth-order harmonic of the Bessel beam is radially limited diffracting and its beamwidth is exactly $1/n$ times that of the fundamental. Many advantages of the Bessel fundamental beam have been demonstrated in the fields of ultrasonic imaging and tissue characterization.[5,3] Here we point out an additional advantage of this beam when it is applied to harmonic imaging due to the nonlinearity of media.[7–9] It is known theoretically and experimentally that for conventional ultrasonic beams (focused or not), the beamwidth of the nonlinearly generated nth-order harmonic is generally $1/\sqrt{n}$ times that of the fundamental.[13,14] The present analysis indicates that in ultrasonic imaging due to the signal of the nth-order harmonic component, higher resolution can be obtained by using the Bessel beam rather than conventional beams that have the same resolution at the fundamental frequency. It should be noticed that in frequency-dependent attenuating media such as biological soft tissues, harmonics of very high orders are very weak and thus cannot be measured in practice. However, this is true for both limited diffraction Bessel beams and conventional beams.

We must emphasize the validity of Eq. (21) that is derived based on the so-called quasilinear (or successive) approximation method. This equation is not a uniformly accurate expression for the nth-order harmonic component of the Bessel beam. From the perturbation theory, the analysis in this paper is valid when the following inequality is satisfied:

$$\left(\frac{2}{\pi}\right)^{1/2}\frac{\beta(ka)^2}{\alpha}\left(\frac{u_0}{c}\right)\eta^{1/2}<\frac{1}{\sqrt{2}}, \tag{22}$$

where β is the acoustic nonlinearity coefficient of the medium and u_0 is the vibration velocity at the source center. This condition coincides with that obtained in the second harmonic case.[7] This indicates that the nth-order harmonic Bessel beam has a finite depth of field. In fact, the depth of field of the harmonics may be similar to that of the fundamental.

The current analysis is based on the ''ideal'' case under which the aperture of the sound source, i.e., the Bessel beam function, is infinite. In this case, the depth of field of the Bessel fundamental beam is extended into infinity. Accordingly, the depth of field of an nth-order harmonic, as predicted by Eq. (21), is also infinite. In practical applications, however, the aperture sizes of beams are always finite. With a finite aperture, the depth of field[2,4] of a Bessel beam is finite, and is a function of the scaling parameter, the aperture radius, and the wavelength of the beam. Within the depth of field, the property of the fundamental Bessel beam can still be characterized by Eq. (4) and the same may also be true for the nth-order harmonic [Eq. (21)] of the Bessel beam but with a beamwidth that is n times smaller than that of the fundamental. This has been verified numerically in the case of the second harmonic.[8]

Finally, our analysis has ignored the attenuation of sound in media. Although this has little influence on the radial distribution of sound beams, it leads to an exponential decay of the amplitudes in axial direction. In medical applications, higher harmonics have a higher attenuation that may limit the depth of penetration. A further theoretical and experimental study will be conducted for the higher harmonics in attenuating media.

III. CONCLUSION

We have obtained a theoretical expression of an nth-order harmonic component of the Bessel beam. The result shows that an nth-order harmonic of the Bessel beam is also limited diffracting in the radial direction and the main beamwidth is exactly $1/n$ times that of the fundamental component.

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[1] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill, New York, 1941), p. 356.
[2] J. Durnin, J. Opt. Soc. Am. A **4**, 651 (1987).
[3] J.-y. Lu, IEEE Trans. Ultrason. Ferroelectr. Freq. Control **44**, 839 (1997), and references therein.
[4] J. Durnin, J. J. Miceli, and J. H. Eberly, Phys. Rev. Lett. **58**, 1499 (1987).
[5] J.-y. Lu and J. F. Greenleaf, IEEE Trans. Ultrason. Ferroelectr. Freq. Control **37**, 438 (1990).
[6] J.-y. Lu and J. F. Greenleaf, IEEE Trans. Ultrason. Ferroelectr. Freq. Control **39**, 19 (1992).
[7] D. Ding and Z. Lu, Appl. Phys. Lett. **68**, 608 (1996); *ibid.*, **71**, 723 (1997).
[8] J. Synnevag and S. Holm, IEEE Ultrason. Symp., Sendai, Oct. 1998.
[9] M. A. Averkiou, D. N. Roundhill, and J. E. Powers, IEEE Ultrason. Symp., Toronto, Oct. 1997.
[10] T. Wulle and S. Herminghaus, Phys. Rev. Lett. **70**, 1401 (1993).
[11] S. Aanonsen, T. Barkve, J. N. Tjøtta, and S. Tjøtta, J. Acoust. Soc. Am. **75**, 749 (1984).
[12] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables* (Dover, New York, 1964).
[13] L. Germain and J. D. N. Cheeke, J. Acoust. Soc. Am. **83**, 942 (1988).
[14] D. Rugar, J. Appl. Phys. **56**, 1338 (1984).