

Second-harmonic generation of the n th-order Bessel beam

Desheng Ding

Department of Electronic Engineering, Southeast University, Nanjing 210096, People's Republic of China

Jian-yu Lu

Ultrasound Laboratory, Department of Bioengineering, The University of Toledo, Toledo, Ohio 43606-3390

(Received 3 May 1999)

We investigate the second-harmonic generation of the n th-order Bessel beam in the nonlinear medium. The analysis is based on the Khokhlov-Zabolotskaya-Kuznetsov wave equation under the second-order approximation in nonlinear acoustics. The theory indicates that for an n th-order Bessel beam, the second-harmonic beam is nearly diffraction-free in the radial direction and behaves as a Bessel beam of the order $2n$, and that the axial pressure amplitude is proportional to the square root of propagation distance. A variety of applications in many fields of nonlinear acoustics and nonlinear optics is expected.

PACS number(s): 42.25.-p, 42.65.Ky

I. INTRODUCTION

The first to discover the Bessel beam solution of the wave equation was Stratton [1]. Forty-six years later, Durnin rediscovered this beam and named it a nondiffracting (or diffraction-free) beam [2]. Because all practical beams are subject to diffraction, a new term, "limited diffraction beams," is used to avoid the controversy of Durnin's terminologies [12]. Durnin *et al.* have also verified the theory with an optical experiment and pointed out potential applications of these new beams [2,3]. Since then, Bessel beams and some of more general limited diffraction beams, such as X wave beams, have been widely investigated in many fields of acoustics, optics, and the relevant science of physics [4–14].

An n th-order Bessel beam is characterized for all values of the propagation distance z by a transverse field distribution proportional to $J_n(\alpha' r)$ with an azimuthal variation of $\exp(in\phi)$, where r is the transverse coordinate, α' is the transverse wave number of the Bessel beam, and J_n denotes the n th-order Bessel function of the first kind [5,11]. These beams are special cases of limited diffraction beams given by Durnin, or solutions to the wave equation in cylindrical coordinates studied in detail by Stratton. In fact, the n th-order Bessel beam is the basis mode of a general limited diffraction beam, that is, an arbitrary limited diffraction beam is the linear superposition of the Bessel beams [8]. Theoretically, the Bessel beam has infinite aperture (and energy), and can travel to infinity without any spreading. For a physically realizable system, however, the aperture of a Bessel beam is always finite. Even so, the beam has a very large depth of field where the beam profile basically remains a Bessel function distribution. The zeroth-order Bessel beam has been applied in medical ultrasonic imaging and tissue identification, showing many advantages on improving the quality of image [8,9]. It may also have potential applications in harmonic imaging which has been recently developed [13,14]. In addition, the dispersion feature of the J_0 beam has been demonstrated to be applicable in nonlinear optics, where this beam can be viewed as a light beam with a tunable wavelength [10].

In this paper, we report a general result on the second

harmonic generation of the n th-order Bessel beam in a nonlinear medium. Our theory shows that the second harmonics of both the J_0 beam [13] and the J_n beam are nearly radially limited diffraction. Generally, for a Bessel beam $J_n(\alpha\xi)\exp(in\phi)$, the nonlinearly generated second-harmonic beam has the field distribution $J_{2n}(2\alpha\xi)\exp(i2n\phi)$. The second harmonic amplitude along the propagation direction of the beam is approximately proportional to the square root of the propagation distance, z .

II. THEORY AND RESULTS

We begin our analysis with the linearized and quasilinear solutions of the Khokhlov-Zabolotskaya-Kuznetsov (KZK) equation in nonlinear acoustics [15–17]. [Because of the similarity of the solutions with those in nonlinear optics, the analysis here is also applicable to the phase-matched second harmonic generation of an intense Bessel (light) beam in the optical-nonlinear medium.] Suppose that a sound source, with an angular frequency ω and a characteristic radius a , oscillates harmonically in time. Furthermore, we assume for simplicity that the absorption and dispersion of the medium are ignored [18] and the beam field has the form (boundary condition) $f_n(\xi')\exp(in\phi')$ on the source. In this case, the radial and azimuthal variables in the field integrals are separable, and the linearized solution for the fundamental pressure field can be expressed in a complex-valued form:

$$\begin{aligned} \bar{q}_1(\xi, \eta, \phi) &= \bar{q}_1(\xi, \eta)\exp[in(\phi + \pi/2)] \\ &= \exp[in(\phi + \pi/2)] \frac{2}{i\eta} \int_{\xi'=0}^{\infty} \int_{\phi'=0}^{2\pi} \\ &\quad \times \exp\left(i\frac{\xi^2 + \xi'^2}{\eta}\right) J_n\left(\frac{2\xi\xi'}{\eta}\right) f_n(\xi')\xi' d\xi', \end{aligned} \quad (1)$$

and the quasilinear solution for the second-harmonic component is

$$\begin{aligned}
\bar{q}_2(\xi, \eta, \phi) &= \bar{q}_2(\xi, \eta) \exp[i2n(\phi + \pi/2)] \\
&= \exp[i2n(\phi + \pi/2)] \frac{1}{2} \int_{\eta'=0}^{\eta} \int_{\xi'=0}^{\infty} \frac{\xi'}{\eta - \eta'} \\
&\quad \times \exp\left(i2 \frac{\xi^2 + \xi'^2}{\eta - \eta'}\right) J_{2n}\left(\frac{4\xi\xi'}{\eta - \eta'}\right) \bar{q}_1^2(\xi', \eta') \\
&\quad \times d\xi' d\eta', \tag{2}
\end{aligned}$$

where $\xi = r/a$ and $\eta = 2z/ka^2$ are the radially and axially dimensionless coordinates, and $k = \omega/c$ is the wave number at the fundamental frequency. Correspondingly, the notations r and z denote the radial and axial coordinates. Here, the characteristic amplitude p_0 of the fundamental is not written out in Eq. (1), nor in Eq. (2) for the second-harmonic amplitude in terms of p_0^2 and the acoustic nonlinearity coefficient of the medium. In the derivation of Eq. (2), it has been assumed that the second harmonic is not generated at the source plane. Equations (1) and (2), where the factors $\exp(-i\tau)$ and $\exp(-i2\tau)$ are suppressed and $\tau = \omega t - kz$, may be considered as the complex-valued pressure amplitudes in dimensionless form for the fundamental and second harmonic components, respectively. When $n=0$, these two equations are the fundamental and second harmonic field of an axially symmetric source [13,17].

We now assume that the sound beam on the source is given by

$$f_n(\xi') = J_n(\alpha\xi'), \tag{3}$$

which produces the n th-order Bessel function of the first kind, where $\alpha = \alpha' a$ is a scaling parameter of J_n . According to Eq. (1) and using the following formulas (4) [19]:

$$\begin{aligned}
&\int_0^{\infty} J_n(\alpha t) J_n(\beta t) e^{\pm i\gamma^2 t^2} t dt \\
&= \pm \frac{i}{2} \gamma^{-2} \exp\left[\mp \frac{i}{4} \gamma^{-2} (\alpha^2 + \beta^2)\right] J_n\left(\frac{1}{2} \alpha\beta\gamma^{-2}\right), \tag{4}
\end{aligned}$$

we obtain the n th-order Bessel fundamental beam,

$$\bar{q}_1(\xi, \eta) = J_n(\alpha\xi) \exp\left(-\frac{i}{4} \alpha^2 \eta\right). \tag{5}$$

Compared with the original form Ref. [2], Eq. (5) has an additional exponential term with an imaginary argument. It is due to the fact that the paraxial approximation has been used in the derivation of the integral representations [Eqs. (1) and (2)] of the field. This approximation is valid in most cases of the propagation of ultrasound (and light) beams.

Substituting Eq. (5) into Eq. (2), we have the second harmonic of the J_n Bessel beam

$$\begin{aligned}
\bar{q}_2(\xi, \eta) &= \frac{1}{2} \int_{\eta'=0}^{\eta} \int_{\xi'=0}^{\infty} \frac{\xi'}{\eta - \eta'} \\
&\quad \times \exp\left(i2 \frac{\xi^2 + \xi'^2}{\eta - \eta'}\right) J_{2n}\left(\frac{4\xi\xi'}{\eta - \eta'}\right) J_n^2(\alpha\xi') \\
&\quad \times \exp\left(-\frac{i\alpha^2 \eta'}{2}\right) d\xi' d\eta'. \tag{6}
\end{aligned}$$

With

$$J_n^2(z) = \frac{2}{\pi} \int_0^{\pi/2} J_{2n}(2z \sin t) dt, \tag{7}$$

we transform Eq. (6) into a triple integral,

$$\begin{aligned}
\bar{q}_2(\xi, \eta) &= \frac{1}{\pi} \int_{\eta'=0}^{\eta} \int_{\xi'=0}^{\infty} \int_{t=0}^{\pi/2} \frac{\xi'}{\eta - \eta'} \\
&\quad \times \exp\left(i2 \frac{\xi^2 + \xi'^2}{\eta - \eta'}\right) J_{2n}\left(\frac{4\xi\xi'}{\eta - \eta'}\right) \\
&\quad \times J_{2n}(2\alpha\xi' \sin t) \exp\left(-\frac{i}{2} \alpha^2 \eta'\right) d\xi' d\eta' dt. \tag{8}
\end{aligned}$$

Using Eq. (4) again and integrating Eq. (8) with respect to ξ' , we then have,

$$\begin{aligned}
\bar{q}_2(\xi, \eta) &= \frac{i}{4\pi} \int_{\eta'=0}^{\eta} \int_{t=0}^{\pi/2} \\
&\quad \times \exp\left(-\frac{i\alpha^2 \eta'}{2} \sin^2 t\right) J_{2n}(2\alpha\xi \sin t) \\
&\quad \times \exp\left[-\frac{i\alpha^2 \eta'}{2} \cos^2 t\right] d\eta' dt. \tag{9}
\end{aligned}$$

It is easy to carry out the above integral about η' and reduce Eq. (9) to

$$\begin{aligned}
\bar{q}_2(\xi, \eta) &= -\frac{1}{2\pi\alpha^2} \exp\left(-\frac{i\alpha^2 \eta}{2}\right) \\
&\quad \times \int_{t=0}^{\pi/2} J_{2n}(2\alpha\xi \sin t) \\
&\quad \times \left[1 - \exp\left(\frac{i\alpha^2 \eta}{2} \cos^2 t\right)\right] \frac{1}{\cos^2 t} dt. \tag{10}
\end{aligned}$$

Equation (10) is an exact expression for the second-harmonic beam of J_n under the second-order approximation, and it can be analytically expressed in terms of generalized hypergeometric functions and Bessel functions (see the Appendix), with the help of a relatively complicated procedure. We prefer to apply an alternative approximation, semianalytical but much simpler, to show the properties of the second harmonic beam given by Eq. (10). Before doing so, heuristically we present a physical explanation of Eq. (10), from the viewpoint of the angular spectrum. The second harmonic of an n th-order Bessel beam may be understood as a linear super-

position of all the $2n$ th-order Bessel beams $J_{2n}(2\alpha\xi \sin t)$ with the transverse wave number from 0 to 2α (correspondingly, the limit of integration is from 0 to $\pi/2$), and with the complex amplitude (or proportional to) $f(t)$ of the angular spectrum, where the function $f(t)$ is given by

$$f(t) = [1 - \exp(iz_1 \cos^2 t)] / \cos^2 t, \quad (11)$$

with $z_1 = \alpha^2 \eta / 2$. It is easy to show that for a sufficiently large z_1 , the real and imaginary parts of this function are extremely similar to the δ function $\delta(t - \pi/2)$. This fact implies that when the second harmonic of the n th-order Bessel beam is generated in the far range from the source (from the following analysis, this range is determined by $z_1 \geq 2\pi$), the ‘‘component’’ $J_{2n}(2\alpha\xi)$ with the transverse wave number 2α is predominant. The numerical integration of the function (11) about t shows that

$$\int_0^{\pi/2} f(t) dt = \left(\frac{\pi}{2}\right)^{1/2} \left(\frac{\alpha^2 \eta}{2}\right)^{1/2} (1-i) \quad (12)$$

is a surprisingly good result when $z_1 \geq 2\pi$. From the property of the δ function, the J_n second-harmonic beam is then well approximated by

$$\bar{q}_2(\xi, \eta) = -\frac{1-i}{4\pi^{1/2}\alpha} \eta^{1/2} J_{2n}(2\alpha\xi) \exp\left(-\frac{i}{2}\alpha^2 \eta\right), \quad (13)$$

in the region of $1/2\alpha^2 \eta \geq 2\pi$. When generated in the range of $0 \leq \frac{1}{2}\alpha^2 \eta < \pi/2$, extremely close to the source, the second-harmonic component has the form

$$\bar{q}_2(\xi, \eta) = \frac{i\eta}{8} J_n^2(\alpha\xi) \exp\left(-\frac{i\alpha^2 \eta}{2}\right). \quad (14)$$

Between these two regions, i.e., when $\pi/2 \leq \frac{1}{2}\alpha^2 \eta < 2\pi$, the radial behavior of the second-harmonic component deviates slightly from the prediction for Eqs. (13) and (14). Although differences exist in the latter two ranges, it is reasonable to think that Eq. (13) reveals the characteristics of the second harmonic of the J_n beam, since the second harmonic associated with the Bessel function $J_{2n}(2\alpha\xi)$ is propagated in nearly the entire region of the beam if the assumption of infinite apertures is considered. From the above analysis, we therefore see that the generated second harmonic of an arbitrary n th-order Bessel beam has the following characteristics:

(i) Like the n th-order Bessel fundamental beam, the second-harmonic beam is also limited diffraction (or more exactly speaking, nearly radially limited diffraction), and the pressure amplitude is simply proportional to the squared root of the propagation distance.

(ii) The second-harmonic component of the n th-order Bessel beam has the Bessel function distribution of the order $2n$, and the corresponding scaling parameter is exactly twice that of the Bessel fundamental beam.

(iii) The n th-order ($n > 0$) Bessel fundamental beam has a spiral wave front dislocation with polarity n at its center [11]. Correspondingly, the second harmonic of this beam is still spiral, with polarity $2n$.

(iv) The zeroth-order Bessel beam is an extremely special case in all Bessel beams. It has a sharp intensity peak at its center. The second-harmonic component of this beam is also a zeroth-order Bessel beam with 2α , and has exactly one-half times the beamwidth of the fundamental [13]. In most cases of the conventional beams (focused or not, ultrasonic or optical), the beamwidth of the nonlinearly generated second harmonic is generally $1/\sqrt{2}$ times that of the fundamental. This has been experimentally verified and theoretically predicted [20,21].

In addition, it must be emphasized that Eqs. (10) and (13) are derived under the quasilinear approximation. From the perturbation theory, these solutions are valid when the following inequality is satisfied:

$$\left(\frac{2}{\pi}\right)^{1/2} \frac{\beta(ka)^2}{\alpha} \left(\frac{u_0}{c}\right) \eta^{1/2} < C. \quad (15)$$

Here, $C \sim 0(1)$, β is the acoustic nonlinear coefficient of medium, u_0 is the vibration velocity at the source center, and u_0/c is the acoustic Mach number.

III. DISCUSSION

It should also be noted that our present analysis is based on the assumption of an infinite aperture. The aperture of practical Bessel (limited diffraction) beams is always finite and the ‘‘Bessel function’’ characteristic of such beams remains only within the depth of field. However, we may point out that the conclusions here will hold in the case of finite aperture, as long as many lobes of the Bessel function are contained within the effective aperture of the Bessel beam. The detailed analysis of this case will be given separately.

Many practical applications of the Bessel beams have been demonstrated in both acoustics (medical ultrasonic imaging and tissue identification) and optics (precision alignment). It may be expected that the higher-order Bessel beams also have potential applications in nonlinear optics and nonlinear acoustics [7,10]. In particular, we want to point out that the J_0 Bessel beam may be very useful for harmonic imaging, where both narrow beamwidth and large field depth are required not only for the fundamental ultrasonic beam but also for the nonlinearly generated second-harmonic component. An extra improvement of imaging resolution may be obtained by using this beam rather than conventional beams as indicated by the property (iv) above.

IV. CONCLUSION

Our theoretical analysis shows that the second harmonic of the n th-order Bessel beam is nearly limited diffraction, with the Bessel function distribution of the order $2n$ and the scaling 2α , and that the amplitude along the beam axis is simply related to the square root of the propagation distance. This result is not only interesting from a fundamental point of view, where the n th-order Bessel beam is ‘‘basis’’ modes of the limited diffraction beam (i.e., any limited diffraction beams are a linear superposition of the Bessel beams, for example, X wave beams) [8], it also offers valuable insight into the study of the nonlinear characteristics of more general limited diffraction beams.

ACKNOWLEDGMENTS

This work was supported in part by the National Natural Science Foundation of China via Research Grant No. C-A040502-19904-003, and by Grant No. HL60301 from the U.S. National Institute of Health.

APPENDIX

An exact analytical expression of the integral in Eq. (10), denoted by $X(\xi, \eta)$, is in terms of generalized hypergeometric functions ${}_2F_2$:

$$X(\xi, \eta) = \frac{(-1)^{n+1} \pi z}{2} \sum_{m=-\infty}^{\infty} J_m(\alpha \xi) J_{m+2n}(\alpha \xi) F_{m,n}(z), \quad (\text{A1})$$

where

$$F_{m,n}(z) = \frac{z^\mu}{2^{2\mu} (\mu+1)!} {}_2F_2\left(\mu + \frac{1}{2}, \mu+1; \mu+2, 2\mu+1; z\right), \quad (\text{A2})$$

with $\mu = |m+n|$ and $z = \frac{1}{2} i \alpha^2 \eta$. In fact, Eq. (14) corresponds to the term of $m = -n$ in Eqs. (A1) and (A2). The approximate solution (13) can be analytically obtained from this general form of (A1) and (A2), by using the asymptotic expansion of ${}_2F_2$ for large argument $|z|$ and the sum theorem of Bessel functions [19].

-
- [1] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill, New York, 1941), p. 356.
- [2] J. Durnin, *J. Opt. Soc. Am. A* **4**, 651 (1987).
- [3] J. Durnin, J. J. Miceli, Jr., and J. H. Eberly, *Phys. Rev. Lett.* **58**, 1499 (1987).
- [4] F. Gori, G. Guattari, and C. Padovani, *Opt. Commun.* **64**, 491 (1987).
- [5] A. Vasara, J. Turunen, and A. T. Friberg, *J. Opt. Soc. Am. A* **6**, 1748 (1989).
- [6] D. K. Hsu, F. J. Margetan, and D. O. Thompson, *Appl. Phys. Lett.* **55**, 2066 (1989).
- [7] R. M. Herman and T. A. Wiggins, *J. Opt. Soc. Am. A* **8**, 932 (1991).
- [8] J.-y. Lu, *IEEE Trans. Ultrason. Ferroelectr. Freq. Control* **44**, 181 (1997).
- [9] J.-y. Lu and J. F. Greenleaf, *Ultrasound Med. Biol.* **17**, 265 (1991).
- [10] T. Wulle and S. Herminghaus, *Phys. Rev. Lett.* **70**, 1401 (1993).
- [11] C. Paterson and R. Smith, *Opt. Commun.* **124**, 121 (1996).
- [12] J.-y. Lu, *IEEE Trans. Ultrason. Ferroelectr. Freq. Control* **45**, 84 (1998), and references therein.
- [13] D. Ding and Z. Lu, *Appl. Phys. Lett.* **68**, 608 (1996).
- [14] J. Synnevag and S. Holm (unpublished).
- [15] E. A. Zabolotskaya and R. V. Khokhlov, *Akust. Zh.* **15**, 40 (1969) [*Sov. Phys. Acoust.* **15**, 35 (1969)].
- [16] V. P. Kuznetsov, *Akust. Zh.* **16**, 548 (1969) [*Sov. Phys. Acoust.* **16**, 467 (1971)].
- [17] S. Aanonsen, T. Barkve, J. N. Tjøtta, and S. Tjøtta, *J. Acoust. Soc. Am.* **75**, 749 (1984).
- [18] M. F. Hamilton and F. H. Fenlon, *J. Acoust. Soc. Am.* **76**, 1474 (1984). Most acoustical media exhibit weak dispersion. The most common cause of dispersion is relaxation, for which the dispersivity is measured by $m = (c_\infty^2 - c_0^2)/c_0^2$, where c_0 and c_∞ are the sound speeds at zero and infinite frequency, respectively. For example, in sea water the value of m is in general less than 10^{-4} . In various liquid polymers, m can be as large as 0.1. Therefore, in most cases there has been very little effect of dispersion in the propagation of sound beams, compared with the nonlinearity of media. On the contrary, the dispersion in nonlinear optics cannot be generally ignored because of the strong dispersivity of most optical dielectrics [see, for example, N. Bloembergen, *Nonlinear Optics* (Benjamin, New York, 1982)]. But the present analytical result is still true to such an extreme case, i.e., the phase-matched SHG of Bessel beams in the optical-nonlinear medium.
- [19] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic, New York, 1980).
- [20] L. Germain and J. D. N. Cheeke, *J. Acoust. Soc. Am.* **83**, 942 (1988).
- [21] D. Rugar, *J. Appl. Phys.* **56**, 1338 (1984).