An X Wave Transform

Jian-yu Lu, *Senior Member, IEEE*, and Anjun Liu

***Abstract*—Limited diffraction beams such as X waves can propagate to an infinite distance without spreading if they are produced with an infinite aperture and energy. In practice, when the aperture and energy are finite, these beams have a large depth of field with only limited diffraction. Because of this property, limited diffraction beams could have applications in medical imaging, tissue characterization, blood flow velocity vector imaging, nondestructive evaluation (NDE) of materials, communications, and other areas such as optics and electromagnetics. In this paper, a new transform, called X wave transform, is developed. In the transform, any well behaved solutions to the isotropic-homogeneous wave equation or limited diffraction beams can be expanded using X waves as basis functions. The coefficients of the expansions can be calculated with the properties that X waves are orthogonal. Examples are given to demonstrate the efficacy of the X wave transform. The X wave transform reveals an intrinsic relationship between any well behaved solutions to the wave equation and X waves, including limited diffraction beams. This provides a theoretical foundation to develop new limited diffraction beams or solutions to the wave equation that may have practical usefulness.**

I. Introduction

Limited diffraction beams were first discovered by Stratton in 1941 [1]; he obtained a Bessel beam solution to the isotropic-homogenous wave equation. Forty-six years later, Durnin [2] and Durnin *et al.* [3] produced these beams approximately with an optical experiment. Theoretically, limited diffraction beams can propagate to an infinite distance without spreading if they are produced with an infinite aperture and energy. In practice, when the aperture and energy are always finite, these beams have a large depth of field (stay in focus over a large depth of interest). Because of this advantage, these beams and other related beams such as localized waves [4]–[8] have been studied extensively for medical imaging [9]–[12], tissue property identification [13], blood flow velocity vector measurement [14], [15], nondestructive evaluation of materials [16], communications [17], and other areas such as optics [2], [3] and electromagnetics [18].

In the early 1990s, families of generalized limited diffraction solutions to the wave equation were discovered [19]–[27]. One subset of these solutions is called X wave because the shape of the waves resemble the letter "X" in a plane along the wave axis [20], [21]. X waves are polychromatic waves and, in theory, can propagate at the speed $c_1 = c/\cos\zeta$, to an infinite distance without spreading in both transverse and axial direction (similar to a wave package), in which c is the speed of sound in a medium or the speed of light in vacuum, and ζ is an Axicon angle [28]–[30]. In practice, nearly exact X waves can be realized with either broadband or band-limited radiators over deep depth of field [21]. Recently, these waves were proposed to construct limited diffraction array beams [31], [32] for ultrahigh frame rate (up to 3750 frames or volumes/s for biological soft tissues at a depth of about 200 mm) two-dimensional (2-D) and three-dimensional (3-D) imaging [33]–[36], and for two-way (transmit-receive) dynamic focusing imaging without montage [37]. X waves and limited diffraction beams also have been studied by other investigators [38]–[42].

From the study of limited diffraction beams in 1997, Lu [32] has developed a method to construct limited diffraction beams of desired properties using X waves or Bessel beams as basis functions. This method poses interesting questions: can any waves that are well behaved solutions (nonsingular, continuous in both magnitude and phase, and derivable) to the wave equation be expressed by a linear superposition of X waves (X wave expansion)? If they can, how does one get the coefficients of the expansion? In this paper, we will develop a new transform, called an X wave transform to answer these questions. To express the waves that are well behaved solutions to the wave equation with X waves, general solutions first are derived with the separation of variables method [43], then converted to an expression using X waves as basis functions. The coefficients of the expression then are obtained using the orthogonality and completeness [44] of the limited diffraction portion of an Axicon beam [28]–[30] that is also the basis function of X waves. The X wave expansions and the formulas for calculating the coefficients compose the X wave transform. The X wave transform is significant because it establishes a relationship between any well behaved solutions, including limited diffraction solutions to the wave equation and X waves. It generalizes the theory of limited diffraction beams and may help to find more limited diffraction beams or solutions to wave equations that may have practical applications [32].

In Section II, we first will derive the X wave expansions for any well behaved solutions to the wave equation. Then, the coefficients of the expansions will be obtained. Three examples are given to demonstrate the efficacy of the X wave transform. In Sections III and IV, we will give a brief discussion and conclusion.

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The authors are with the Ultrasound Laboratory, Department of Bioengineering, The University of Toledo, Toledo, OH 43606 (e-mail: jilu@eng.utoledo.edu).

II. Theory

A. Solutions to Wave Equation and X Wave Expansions

1. X Waves: A three-dimensional scalar wave equation in a cylindrical coordinates for source-free, loss-less, and isotropic-homogeneous media is given by:

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\Phi=0, \tag{1}$$

where $\Phi=\Phi(\vec{r},t)$ denotes acoustic pressure or Hertz potential at a spatial point $\vec{r}=(r,\phi,z)$ and time t, where $r=\sqrt{x^2+y^2}$ is a radial distance, x and y are variables in rectangular coordinates, ϕ is azimuthal angle, z is the axial axis, and c is the speed of sound in the medium or the speed of light in a vacuum.

One of the families of solutions of (1) is an nth-order X wave [20] that is a linear superposition of the limited diffraction portion of an Axicon beam [30]:

$$\begin{aligned}\Phi_{X_{n,\zeta}}(\vec{r},t) &= \Phi_{X_{n,\zeta}}(r,\phi,\pm(z\pm c_1 t))\\ &= e^{in\phi}\int_0^\infty T_{n,\zeta}(k)J_n(kr\sin\zeta)e^{\pm ik\cos\zeta(z\pm c_1t)}dk,\\ &\qquad (n=0,\,\pm1,\,\pm2,\,\cdots),\end{aligned} \tag{2}$$

where $c_1=c/\cos\zeta$ is both the group and phase velocity of the X wave ($d\omega/dk_z=\omega/k_z=c_1\ge c$, where $k_z=k\cos\zeta$), $k=\omega/c$ is the wave number, $\omega=2\pi f$ is the angular frequency, f is the temporal frequency, $\zeta(0\le\zeta<\pi/2)$ is the Axicon angle [28]–[30] of the X wave, $J_n(\cdot)$ is the nth-order Bessel function of the first kind, n is termed the "order" of the waves, $T_{n,\zeta}(k)=B_{n,\zeta}(k)e^{-ka_0}$, where a_0 is a constant that determines the fall-off speed of the high-frequency components of X waves and $B_{n,\zeta}(k)$ is any well behaved function that could represent the transfer function of a practical acoustic transducer or electromagnetic antenna for a given n and ζ.

Notice that, in (2), the "$\pm$" in front of c_1 represent backward and forward propagating waves along the z axis, respectively. The "$\pm$" in front of ik represents the positive and negative directions, respectively, of both the axial distance, z, and time, t. With the negative sign, we have:

$$\Phi_{X_{n,\zeta}}(r,\phi,-(z\pm c_1t))=\Phi_{X_{n,\zeta}}(r,\phi,(-z)\pm c_1(-t)), \tag{3}$$

which is similar to that of a positive sign, but the variables are in a reverse direction. For the simplicity of the presentation, in the following, we consider only the forward propagating waves and the positive direction of both the axial distance and time. All results will be the same for the backward propagating waves and for the case of a reverse direction of both the axial distance and time.

2. Arbitrary Solutions: From the mathematical physics, a general solution to the wave equation (1) can be obtained with a separation of variables method [43], [45]. Assuming

$$\Phi(\vec{r},t)=\mathrm{U}(\vec{r})\mathrm{P}(t) \tag{4}$$

and inserting (4) into (1), one obtains the following two equations:

$$\mathrm{P}''(t)+k^2c^2\mathrm{P}(t)=0 \tag{5}$$

and

$$\nabla^2\mathrm{U}(\vec{r})+k^2\mathrm{U}(\vec{r})=0, \tag{6}$$

where "$''$" represents second derivative with respect to the variable, t. A well behaved solution to (5) is given by:

$$\mathrm{P}(t)=C_Ae^{ikct}+C_Be^{-ikct}, \tag{7}$$

where $k\ge 0$ is the wave number, and C_A and C_B are arbitrary complex constants. Let

$$\mathrm{U}(\vec{r})=\mathrm{R}(r)\Pi(\phi)\mathrm{Z}(z), \tag{8}$$

we have the solutions to (6) [43], [45] [see (9) on top of next page]: where $n=0,\pm1,\pm2,\ldots$, are integers, $I_n(\cdot)$ is a hyperbolic (complex-argument) Bessel function of the first kind (page 1259 of [45]), C_C, C_D, C_E, and C_F are arbitrary complex constants, and μ, $k=\omega/c$, and h are real constants. $h^2=-\mu$ when $\mu<0$.

When $\mu=0$, solving (6) directly by the separation of variables (8), we have:

$$\mathrm{Z}(z)=C_E+C_Dz, \tag{10}$$

which is infinity when both $C_D\ne0$ and $z\to\pm\infty$. For $Z(z)$ to remain finite as $z\to\pm\infty$, $C_D\equiv0$. In this case, $\Phi(\vec{r},t)$ in (4) will not be a function of z when $\mu=0$. A similar thing happens in (5) too, i.e., $\Phi(\vec{r},t)$ is not a function of t if $k=0$.

We are interested in well behaved solutions that can be approximated with a physical system. This means that only some rows of (9) should be retained. For the first row in (9), if $z\to\pm\infty$ and $C_E\ne0$ or $C_F\ne0$, $U(\vec{r})=\infty$. Such a wave is not physically realizable and should be discarded. Similarly, the third row in (9) is infinity when $r\to\infty$ and also should be discarded. In addition, because the integer, n, assumes both positive and negative values by definition, only one of the terms, C_C and C_D, is needed and thus one can set $C_D\equiv0$. Therefore, (9) is simplified to:

$$\mathrm{U}(\vec{r})=C_Ce^{in\phi}(C_Ee^{ihz}+C_Fe^{-ihz})J_n(\sqrt{k^2-h^2}r). \tag{11}$$

Let $h=k\cos\zeta$, where $0\le\zeta<\pi/2$ is a real constant, we have $\sqrt{k^2-h^2}=k\sin\zeta$. Combining (7) and (11) with (4), one obtains the characteristic solution (eigen solution) to the wave equation (1):

$$\begin{aligned}\Phi_{n,k,\zeta}(r,\phi,z,t) &= (C_Ae^{ikct}+C_Be^{-ikct})\\ &\times C_Ce^{in\phi}(C_Ee^{ikz\cos\zeta}+C_Fe^{-ikz\cos\zeta})J_n(kr\sin\zeta).\end{aligned} \tag{12}$$

$$\mathrm{U}(\vec{r}) = \begin{cases} (C_C e^{in\phi} + C_D e^{-in\phi})\,(C_E e^{\sqrt{\mu}z} + C_F e^{-\sqrt{\mu}z})\,J_n(\sqrt{k^2+\mu}r),\ \mu > 0 \\ (C_C e^{in\phi} + C_D e^{-in\phi})\,(C_E e^{ihz} + C_F e^{-ihz})\,J_n(\sqrt{k^2-h^2}r),\ \ \mu < 0 \text{ and } k^2 \ge h^2, \\ (C_C e^{in\phi} + C_D e^{-in\phi})\,(C_E e^{ihz} + C_F e^{-ihz})\,I_n(\sqrt{h^2-k^2}r),\ \ \mu < 0 \text{ and } k^2 < h^2 \end{cases} \tag{9}$$

As discussed before, if the backward propagating waves and the waves in the reverse directions of both the axial distance and time are ignored, (12) is further simplified:

$$\begin{aligned}\Phi_{n,k,\zeta}(r,\phi,z,t) &= \Phi_{n,k,\zeta}(r,\phi,z-c_1t)\\ &= T_{n,\zeta}(k)\Phi_{A_{n,k,\zeta}}(r,\phi,z-c_1t),\end{aligned} \tag{13}$$

where $c_1 = c/\cos\zeta$ and

$$\Phi_{A_{n,k,\zeta}}(r,\phi,z-c_1t) = e^{in\phi}J_n(kr\sin\zeta)e^{ik\cos\zeta(z-c_1t)} \tag{14}$$

is the limited diffraction portion of the Axicon beam [30] and $T_{n,\zeta}(k) = C_C C_B C_E$.

The general solution to the wave equation is a superposition of (13) over the free parameters (eigenvalues), n, ζ, and k:

$$\begin{aligned}\Phi(\vec{r},t) &= \sum_{n=-\infty}^{\infty}\int_0^{\pi/2} d\zeta \int_0^{\infty} dk T_{n,\zeta}(k)\Phi_{A_{n,k,\zeta}}(r,\phi,z-c_1t)\\ &= \sum_{n=-\infty}^{\infty}\int_0^{\pi/2}\left[e^{in\phi}\int_0^{\infty} T_{n,\zeta}(k)J_n(kr\sin\zeta)e^{ik\cos\zeta(z-c_1t)}dk\right]d\zeta\\ &= \sum_{n=-\infty}^{\infty}\int_0^{\pi/2}\Phi_{X_{n,\zeta}}(r,\phi,z-c_1t)d\zeta.\end{aligned} \tag{15}$$

Notice that the term in the square brackets is an nth-order X wave in (2).

3. Special Case for Limited Diffraction Beams: If $T_{n,\zeta'}(k) = T_{n,\zeta}(k)\delta(\zeta'-\zeta)$, where $T_{n,\zeta}(k)$ is equal to $T_{n,\zeta'}(k)$ evaluated at a constant ζ ($0 \le \zeta \le \pi/2$), and $\delta(\cdot)$ is the Dirac-Delta function [45], (15) represents a limited diffraction beam [32]:

$$\begin{aligned}\Phi(r,\phi,z-c_1t) &= \sum_{n=-\infty}^{\infty}\int_0^{\pi/2}\Phi_{X_{n,\zeta'}}(r,\phi,z-c_1't)d\zeta'\\ &= \sum_{n=-\infty}^{\infty} e^{in\phi}\int_0^{\infty} T_{n,\zeta}(k)J_n(kr\sin\zeta)e^{ik\cos\zeta(z-c_1t)}dk\\ &= \sum_{n=-\infty}^{\infty}\Phi_{X_{n,\zeta}}(r,\phi,z-c_1t)\end{aligned} \tag{16}$$

travelling at a constant group and phase velocity, $c_1 = c/\cos\zeta$ ($c_1' = c/\cos\zeta'$). Travelling with the wave, $\Phi(r,\phi,z-c_1t)$, at the speed, c_1, one sees a constant wave pattern.

Therefore, X waves in (2) are the basis functions of all limited diffraction beams (any limited diffraction beams are a superposition of X waves over the index n). A special case is that $T_{n,\zeta}(k) = T(k)$ for all the index n. This means that all the X wave components of a limited diffraction beam, $\Phi(r,\phi,z-c_1t)$, are produced with the same transducer that has a constant transfer function or bandwidth.

B. Coefficients of X Wave Expansions

Eq. (15) and (16) demonstrate that any well behaved solutions, including limited diffraction solutions to the wave equation, can be expanded with X waves. In the following, we will find the coefficients of the expansions. To get the coefficients of well behaved solutions, we must use the limited diffraction portion of an Axicon beam [see (2) and (14)] [30] that is the basis of X waves and is orthogonal and complete [44]. Combined with the X wave expansions, formulas to obtain the coefficients will produce a new transform, called the X wave transform.

Multiplying both sides of (15) with a conjugation of the limited diffraction portion of an Axicon beam (14) [30]:

$$\begin{aligned}&\Phi^*_{A_{n,k,\zeta}}(r,\phi,z-c_1t)\\ &= e^{-in\phi}J_n(kr\sin\zeta)e^{-ik\cos\zeta(z-c_1t)},\end{aligned} \tag{17}$$

where "*" represents a complex conjugation, and integrating the results over the variables, r, ϕ, and t, we have:

$$\begin{aligned}&\int_0^{\infty} rdr\int_{-\pi}^{\pi} d\phi\int_{-\infty}^{\infty} dt\Phi(r,\phi,z,t)\Phi^*_{A_{n,k,\zeta}}(r,\phi,z-c_1t)\\ &= \int_0^{\infty} rdr\int_{-\pi}^{\pi} d\phi\int_{-\infty}^{\infty} dt\ \times\\ &\left[\sum_{n'=-\infty}^{\infty}\int_0^{\pi/2} d\zeta' e^{in'\phi},\int_0^{\infty} dk' T_{n',\zeta'}(k')J_{n'}(k'r\sin\zeta')e^{ik'\cos\zeta'(z-c_1't)}\right]\\ &\times\left[e^{-in\phi}J_n(kr\sin\zeta)e^{-ik\cos\zeta(z-c_1t)}\right].\end{aligned} \tag{18}$$

Rearranging the terms in (18), one obtains:

$$\int_0^\infty rdr \int_{-\pi}^{\pi} d\phi \int_{-\infty}^{\infty} dt\Phi(r,\phi,z,t)\Phi^*_{A_{n,k,\zeta}}(r,\phi,z-c_1t)$$
$$= 2\pi \sum_{n'=-\infty}^{\infty} \left[\int_{-\pi}^{\pi} d\phi e^{i(n'-n)\phi}\right] \int_0^{\pi/2} d\zeta'$$
$$\times \int_{-\infty}^{\infty} dt \frac{1}{2\pi} \int_{-\infty}^{\infty} dk' T_{n',\zeta'}(k')H(k')$$
$$\left[\int_0^\infty rdrJ_{n'}(k'r\sin\zeta')J_n(kr\sin\zeta)\right]$$
$$\times e^{iz(k'\cos\zeta'-k\cos\zeta)}e^{-ik'ct}e^{ikct}, \tag{19}$$

where $H(k')$ is the Heaviside step function [46], [47]:

$$H(k') = \begin{cases} 1, & k' \geq 0 \\ 0, & \text{Otherwise} \end{cases}, \tag{20}$$

and the integrations over k' and t in (19) are an inverse Fourier transform of:

$$\frac{T_{n',\zeta'}(k')H(k')}{c}$$
$$\left[\int_0^\infty rdr\, J_{n'}(k'r\sin\zeta')J_n(kr\sin\zeta)\right] e^{iz(k'\cos\zeta'-k\cos\zeta)}, \tag{21}$$

followed by a Fourier transform, which are cancelled with each other. With (21), (19) is simplified:

$$\int_0^\infty rdr \int_{-\pi}^{\pi} d\phi \int_{-\infty}^{\infty} dt\Phi(r,\phi,z,t)\Phi^*_{A_{n,k,\zeta}}(r,\phi,z-c_1t)$$
$$= \frac{2\pi H(k)}{c} \sum_{n'=-\infty}^{\infty} \left[\int_{-\pi}^{\pi} d\phi e^{i(n'-n)\phi}\right] \int_0^{\pi/2} d\zeta' T_{n',\zeta'}(k)$$
$$\times \left[\int_0^\infty rdrJ_{n'}(kr\sin\zeta')J_n(kr\sin\zeta)\right] e^{ikz(\cos\zeta'-\cos\zeta)}. \tag{22}$$

The first square brackets in (22) are given by:

$$\int_{-\pi}^{\pi} e^{i(n'-n)\phi} d\phi = 2\pi\delta_K(n'-n) = \begin{cases} 2\pi, & \text{If } n'=n \\ 0, & \text{Otherwise} \end{cases}, \tag{23}$$

where $\delta_K(\cdot)$ is the Kronecker-Delta function. The second square brackets in (22) is given by (page 943 of [45]):

$$\int_0^\infty rdr\, J_n(kr\sin\zeta')\, J_n(kr\sin\zeta)$$
$$= \frac{1}{k^2\sqrt{\sin\zeta'\sin\zeta}}\delta(\sin\zeta'-\sin\zeta). \tag{24}$$

Substituting (23) and (24) into (22), we have:

$$\int_0^\infty rdr \int_{-\pi}^{\pi} d\phi \int_{-\infty}^{\infty} dt\Phi(r,\phi,z,t)\Phi^*_{A_{n,k,\zeta}}(r,\phi,z-c_1t)$$
$$= \frac{(2\pi)^2 H(k)}{k^2 c\sin\zeta\cos\zeta} T_{n,\zeta}(k). \tag{25}$$

From (25), the coefficients of the X wave expansion (15) are obtained:

$$T_{n,\zeta}(k) = \frac{k^2 c\sin\zeta\cos\zeta H(k)}{(2\pi)^2}$$
$$\times \int_0^\infty rdr \int_{-\pi}^{\pi} d\phi \int_{-\infty}^{\infty} dt\Phi(r,\phi,z,t)\Phi^*_{A_{n,k,\zeta}}(r,\phi,z-c_1t), \tag{26}$$

where $H(k)$ is used to indicate that k is positive and thus it can be moved from the denominator to the numerator of (26). The (26) is obtained using the properties of the orthogonality and completeness [44] of the limited diffraction portion of an Axicon beam in (14) [30].

A similar formula can be obtained for (16) to calculate the coefficients of the X wave expansion of limited diffraction beams (see the Appendix):

$$T_{n,\zeta}(k) = \frac{cH(k)}{(2\pi)^2 J_n^2(kr\sin\zeta)}$$
$$\times \int_{-\pi}^{\pi} d\phi \int_{-\infty}^{\infty} dt\Phi(r,\phi,z-c_1t)\Phi^*_{A_{n,k,\zeta}}(r,\phi,z-c_1t), \tag{27}$$

where $H(k)$ also is moved from the denominator to the numerator. Notice that there will always be a term, $J_n^2(kr\sin\zeta)$, from the integrations on the right-hand side of (27) to cancel that in the denominator [see (14) and (16)]. This term is a result of the cancellation of integrations over both k and t, and the Kronecker-Delta function (23) due to the integration over ϕ [see (47)–(50) in the Appendix for details].

C. Examples

To demonstrate the efficacy of the X wave transform, the following three examples are given.

1. A Composite X Wave: A composite X wave is a linear combination of two broadband limited diffraction X waves [20] of different Axicon angles:

$$\Phi(r,\phi,z,t)$$
$$= K_A\Phi_{X_A}(r,\phi,z-c_At) + K_B\Phi_{X_B}(r,\phi,z-c_Bt), \tag{28}$$

where K_A and K_B are arbitrary constants,

$$\Phi_{X_A}(r,\phi,z-c_At)$$
$$= \int_0^\infty J_0(kr\sin\zeta_A)e^{ik\cos\zeta_A(z-c_At)}\, dk \tag{29}$$

and

$$\Phi_{X_B}(r, \phi, z - c_B t) = \int_0^\infty J_0(kr \sin\zeta_B) e^{ik\cos\zeta_B(z - c_B t)}\, dk \tag{30}$$

are the first and second X waves, respectively, ζ_A and ζ_B are Axicon angles, and $c_A = c/\cos\zeta_A$ and $c_B = c/\cos\zeta_B$ are both phase and group velocities. If $\zeta_A \neq \zeta_B$, the composite X wave is not a limited diffraction beam because it cannot be written as $\Phi(r, \phi, z - c_1 t)$, where c_1 is a single constant phase and group velocity of the wave.

The coefficients of the expansion of the composite X wave can be calculated with (26) [see (31) on page 1477].

Rearranging the terms in (31), one obtains (32) [page 1477], and the integrations over k' and then t in (32) are an inverse Fourier transform of (33) [page 1477] followed by a Fourier transform, which are cancelled with each other. With (33), (32) is simplified (34) [page 1477]. Considering (23) for the first brackets in (34), we have (35) [page 1477]. Using (24), one obtains (36) [page 1478], where $H^2(k)$ is replaced with $H(k)$ because they have the same meaning. The last equivalence in (36) is due to the selective property of the Dirac-Delta function.

Substituting (36) into the X wave expansion (15), we obtain the composite X wave (28) [see (37) on page 1478]. Therefore, the coefficients obtained in (36) are correct.

2. 4th-Derivative Bowtie X Wave: Bowtie X waves and bowtie Bessel beams have a potential application in imaging over a large depth of field with very low sidelobes [24], [25]. The 4th-derivative bowtie X wave is given by [24], [32],

$$\Phi_{X_{B4}}(r, \phi, z - c_1 t) = \frac{\partial^4}{\partial y^4} \Phi_{X_{0,\zeta}}(r, \phi, z - c_1 t) = \int_0^\infty T'_{0,\zeta}(k) \left[\frac{\partial^4}{\partial y^4} J_0(kr \sin\zeta) \right] e^{ik\cos\zeta(z - c_1 t)} dk, \tag{38}$$

where $T'_{0,\zeta}(k)$ is related to the transfer function of an acoustic transducer or electromagnetic antenna that produces the bowtie X wave. The term in the square brackets can be calculated using the properties of Bessel functions [24], [32], [45],

$$\frac{\partial^4}{\partial y^4} J_0(kr \sin\zeta) = \frac{1}{8}(k \sin\zeta)^4 \times [3J_0(kr\sin\zeta) + 4J_2(kr\sin\zeta)\cos(2\phi) + J_4(kr\sin\zeta)\cos(4\phi)]. \tag{39}$$

Using the X wave expansion of limited diffraction beams (16), the coefficients of the X wave expansion for the 4th-derivative bowtie X wave can be calculated with (27) [see (40) on page 1478]. Inserting (39) into (40), rearranging the terms in (40), and integrating over k' first then over t in a way similar to the treatment in (18)–(22), we have (41) [page 1478]. Integrating over ϕ (23), one obtains (42) [page 1478].

From (42), we obtain the coefficients of the X wave expansion of the 4th-derivative bowtie X wave:

$$T_{n,\zeta}(k) = \begin{cases} \frac{3(k\sin\zeta)^4 H(k)}{8} T'_{0,\zeta}(k), & n = 0 \\ \frac{(k\sin\zeta)^4 H(k)}{4} T'_{0,\zeta}(k), & n = \pm 2 \\ \frac{(k\sin\zeta)^4 H(k)}{16} T'_{0,\zeta}(k), & n = \pm 4 \\ 0, & \text{Otherwise} \end{cases}. \tag{43}$$

Inserting (43) back into (16), we obtain (38), the 4th-derivative bowtie X wave.

3. Limited Diffraction Array Beams: Any well behaved solutions to the wave equations can be expressed as a linear superposition of limited diffraction array beams through a spatial Fourier transform [8], [47]. In the following, we will show that limited diffraction array beams also can be expressed as a linear superposition of X waves through the X wave transform. This means that any well behaved solutions also can be expanded with X waves (see [31], [32] and (5) and (14) of [33] for limited diffraction array beams):

$$\Phi_{Array}(r, \phi, z - c_1 t) = \sum_{n=-\infty}^{\infty} \Phi_{X_{n,\zeta}}(r, \phi, z - c_1 t) = \sum_{n=-\infty}^{\infty} e^{in\phi} \int_0^\infty \left[i^n e^{-in\theta} T(k) \right] J_n(kr\sin\zeta) e^{ik\cos\zeta(z - c_1 t)} dk. \tag{44}$$

Comparing (44) with (16), one obtains the coefficients of the X wave expansion of a limited diffraction array beam:

$$T_{n,\zeta}(k) = i^n e^{-in\theta} T(k), \quad n = 0, \pm 1, \pm 2, \cdots. \tag{45}$$

III. Discussion

We have developed a new transform, called the X wave transform. With this transform, any well behaved solutions to the wave equation can be expanded with X waves. Reversely, the coefficients of the expansion can be calculated with a limited diffraction portion of an Axicon beam [28]–[30] that is the basis of an X wave and is an orthogonal and complete set [44].

From the third example in the previous section, it is clear that limited diffraction array beams can be expanded with X waves [33]. Using a spatial Fourier transform, any well behaved solutions to the wave equation can be expressed by limited diffraction array beams [8], [47]. This indicates that these solutions also can be expanded with X waves. In fact, X waves are superpositions of plane waves (see (2) of [20]) and vice versa [33]. Therefore, like the plane waves, X waves also are basis functions and can be used as a tool to analyze the solutions to the wave equation. The difference between plane waves and X waves is

$$T_{n,\zeta}(k) = \frac{k^2 c \sin\zeta \cos\zeta H(k)}{(2\pi)^2} \times \int_0^\infty r dr \int_{-\pi}^{\pi} d\phi \int_{-\infty}^{\infty} dt \Phi(r,\phi,z,t)\Phi^*_{A_{n,k,\zeta}}(r,\phi,z-c_1 t) = \frac{k^2 c \sin\zeta \cos\zeta H(k)}{(2\pi)^2} \int_0^\infty r dr \int_{-\pi}^{\pi} d\phi \int_{-\infty}^{\infty} dt \times \int_0^\infty dk' \left[K_A J_0(k' r \sin\zeta_A) e^{ik'\cos\zeta_A(z-c_A t)} + K_B J_0(k' r \sin\zeta_B) e^{ik'\cos\zeta_B(z-c_B t)}\right] \times \left[e^{-in\phi} J_n(kr\sin\zeta) e^{-ik\cos\zeta(z-c_1 t)}\right]. \tag{31}$$

$$T_{n,\zeta}(k) = \frac{k^2 c \sin\zeta \cos\zeta H(k)}{(2\pi)^2} 2\pi \left[\int_{-\pi}^{\pi} d\phi e^{-in\phi}\right] \int_{-\infty}^{\infty} dt \frac{1}{2\pi} \int_{-\infty}^{\infty} dk' \times H(k') \int_0^\infty r dr \left[K_A J_0(k' r \sin\zeta_A) e^{ik' z \cos\zeta_A} + K_B J_0(k' r \sin\zeta_B) e^{ik' z \cos\zeta_B}\right] \times J_n(kr\sin\zeta) e^{-ikz\cos\zeta} e^{-ik' ct} e^{ikct}, \tag{32}$$

$$\frac{H(k')}{c} \int_0^\infty r dr \left[K_A J_0(k' r \sin\zeta_A) e^{ik' z \cos\zeta_A} + K_B J_0(k' r \sin\zeta_B) e^{ik' z \cos\zeta_B}\right] \times J_n(kr\sin\zeta) e^{-ikz\cos\zeta}, \tag{33}$$

$$T_{n,\zeta}(k) = \frac{k^2 \sin\zeta \cos\zeta H^2(k)}{(2\pi)^2} 2\pi \left[\int_{-\pi}^{\pi} d\phi e^{-in\phi}\right] \times \int_0^\infty r dr \left[K_A J_0(kr\sin\zeta_A) e^{ikz\cos\zeta_A} + K_B J_0(kr\sin\zeta_B) e^{ikz\cos\zeta_B}\right] \times J_n(kr\sin\zeta) e^{-ikz\cos\zeta}. \tag{34}$$

$$T_{n,\zeta}(k) = k^2 \sin\zeta \cos\zeta H^2(k) \times \int_0^\infty r dr \left[K_A J_0(kr\sin\zeta_A) e^{ikz\cos\zeta_A} + K_B J_0(kr\sin\zeta_B) e^{ikz\cos\zeta_B}\right] \times J_0(kr\sin\zeta) e^{-ikz\cos\zeta} \delta_K(n). \tag{35}$$

$$\begin{aligned}T_{n,\zeta}(k) &= k^2 \sin\zeta \cos\zeta H(k) \\ &\times \left[K_A \frac{\delta(\sin\zeta_A - \sin\zeta)}{k^2\sqrt{\sin\zeta_A \sin\zeta}} e^{ikz\cos\zeta_A} + K_B \frac{\delta(\sin\zeta_B - \sin\zeta)}{k^2\sqrt{\sin\zeta_B \sin\zeta}} e^{ikz\cos\zeta_B}\right] \\ &\times e^{-ikz\cos\zeta}\delta_K(n) \\ &= \cos\zeta H(k)\left[K_A\delta(\sin\zeta_A - \sin\zeta) + K_B\delta(\sin\zeta_B - \sin\zeta)\right]\delta_K(n), \end{aligned} \tag{36}$$

$$\begin{aligned}\Phi(\vec{r},t) &= \sum_{n=-\infty}^{\infty} \int_0^{\pi/2} \left[e^{in\phi} \int_0^{\infty} T_{n,\zeta}(k) J_n(kr\sin\zeta) e^{ik\cos\zeta(z-c_1 t)} dk\right] d\zeta \\ &= \sum_{n=-\infty}^{\infty} \int_0^{\pi/2} d\zeta e^{in\phi} \int_0^{\infty} dk \\ &\times \cos\zeta\left[K_A\delta(\sin\zeta_A - \sin\zeta) + K_B\delta(\sin\zeta_B - \sin\zeta)\right]\delta_K(n) \\ &\times J_n(kr\sin\zeta) e^{ik\cos\zeta(z-c_1 t)} \\ &= \int_0^{\infty} dk \left[K_A J_0(kr\sin\zeta_A) e^{ik\cos\zeta_A(z-c_A t)} + K_B J_0(kr\sin\zeta_B) e^{ik\cos\zeta_B(z-c_B t)}\right]. \end{aligned} \tag{37}$$

$$\begin{aligned}T_{n,\zeta}(k) &= \frac{cH(k)}{(2\pi)^2 J_n^2(kr\sin\zeta)} \\ &\times \int_{-\pi}^{\pi} d\phi \int_{-\infty}^{\infty} dt \Phi_{X_{B4}}(r,\phi,z-c_1 t)\Phi^*_{A_{n,k,\zeta}}(r,\phi,z-c_1 t) \\ &= \frac{cH(k)}{(2\pi)^2 J_n^2(kr\sin\zeta)} \\ &\times \int_{-\pi}^{\pi} d\phi \int_{-\infty}^{\infty} dt \left[\int_0^{\infty} T'_{0,\zeta}(k') \frac{\partial^4}{\partial y^4} J_0(k' r\sin\zeta) e^{ik'\cos\zeta(z-c_1 t)} dk'\right] \\ &\times \left[e^{-in\phi} J_n(kr\sin\zeta) e^{-ik\cos\zeta(z-c_1 t)}\right]. \end{aligned} \tag{40}$$

$$\begin{aligned}T_{n,\zeta}(k) &= \frac{H(k)}{2\pi J_n^2(kr\sin\zeta)} T'_{0,\zeta}(k) \int_{-\pi}^{\pi} d\phi e^{-in\phi} J_n(kr\sin\zeta) \\ &\times \frac{1}{8}(k\sin\zeta)^4\left[3J_0(kr\sin\zeta) + 4J_2(kr\sin\zeta)\cos 2\phi + J_4(kr\sin\zeta)\cos 4\phi\right]. \end{aligned} \tag{41}$$

$$\begin{aligned}T_{n,\zeta}(k) &= \frac{(k\sin\zeta)^4 H(k)}{16\pi J_n^2(kr\sin\zeta)} T'_{0,\zeta}(k) J_n(kr\sin\zeta) \\ &\times \left[6\pi J_0(kr\sin\zeta)\delta_K(n) + 4\pi J_2(kr\sin\zeta)\delta_K(n-2) + 4\pi J_2(kr\sin\zeta)\delta_K(n+2)\right. \\ &\left. + \pi J_4(kr\sin\zeta)\delta_K(n-4) + \pi J_4(kr\sin\zeta)\delta_K(n+4)\right]. \end{aligned} \tag{42}$$

$$\int_{-\pi}^{\pi} d\phi \int_{-\infty}^{\infty} dt \Phi(r,\phi,z-c_1 t)\Phi^*_{A_{n,k,\zeta}}(r,\phi,z-c_1 t)$$
$$= \int_{-\pi}^{\pi} d\phi \int_{-\infty}^{\infty} dt \left[\sum_{n'=-\infty}^{\infty} e^{in'\phi}\int_0^{\infty} dk' T_{n',\zeta}(k')J_{n'}(k'r\sin\zeta)e^{ik'\cos\zeta(z-c_1 t)}\right]$$
$$\times\left[e^{-in\phi}J_n(kr\sin\zeta)e^{-ik\cos\zeta(z-c_1 t)}\right]. \quad (46)$$

$$\int_{-\pi}^{\pi} d\phi \int_{-\infty}^{\infty} dt \Phi(r,\phi,z-c_1 t)\Phi^*_{A_{n,k,\zeta}}(r,\phi,z-c_1 t)$$
$$= \frac{2\pi H(k)}{c}\sum_{n'=-\infty}^{\infty}\left[\int_{-\pi}^{\pi} d\phi e^{i(n'-n)\phi}\right] T_{n',\zeta}(k)J_{n'}(kr\sin\zeta)J_n(kr\sin\zeta). \quad (49)$$

that the latter are localized in addition to being limited diffraction.

A. Conditions

In the derivation of the X wave transform, we have assumed that the solutions to the wave equation are well behaved. These solutions should not be singular and could possibly be produced with a physical device. Physically realizable solutions to the wave equation are of particular interest in real world.

We also have ignored the backward propagating waves and waves that are in the reverse directions of the axial axis, z, and time, t. These components can be easily added to the results ((15) and (16)) if the solutions to be expanded contain them. However, in the real world, backward propagating waves will not exist if there is no boundary (1).

B. Significance

The X wave transform is significant because it generalizes limited diffraction beams studied previously, reveals the intrinsic relationship among these beams, and establishes a relationship between X waves and any well behaved solutions to the wave equation. Because X waves could have many applications in various areas (see Section I), the X wave transform can be used as a tool to explore new solutions to the wave equation and study their usefulness in practice. This has been demonstrated by the examples in [32] in which a method was developed to construct limited diffraction beams of desired properties and of practical applications. In addition, because X waves can be approximated well with physical devices such as an acoustic transducer, electromagnetic antenna, or a laser gun [40] over a large depth of field, realization of other well behaved solutions to the wave equation also could be explored through the X wave transform.

IV. Conclusions

A new transform, called the X wave transform, is developed. This transform establishes a relationship between the X waves and any well behaved solutions, including limited diffraction solutions to the wave equation. Because X waves are localized in space and can be approximately produced with a physical device over a large depth of field, they could have many applications. Therefore, the X wave transform can be used as a tool to explore new limited diffraction beams or new solutions to the wave equations that may have practical applications.

Appendix

For limited diffraction beams, the coefficients of the X wave expansion (16) can be calculated with a simpler formula (27). The derivation of the formula is similar to that in (26) and will be given in this Appendix.

Multiplying both sides of (16) with a conjugation of the limited diffraction portion of an Axicon beam (17) and integrating the results over both the variables, ϕ and t, we have (46). Rearranging the terms in (46), one obtains:

$$\int_{-\pi}^{\pi} d\phi \int_{-\infty}^{\infty} dt \Phi(r,\phi,z-c_1 t)\Phi^*_{A_{n,k,\zeta}}(r,\phi,z-c_1 t)$$
$$= 2\pi\sum_{n'=-\infty}^{\infty}\left[\int_{-\pi}^{\pi} d\phi e^{i(n'-n)\phi}\right]$$
$$\times\int_{-\infty}^{\infty} dt \frac{1}{2\pi}\int_{-\infty}^{\infty} dk' T_{n',\zeta}(k')H(k')J_{n'}(k'r\sin\zeta)J_n(kr\sin\zeta)$$
$$\times e^{iz\cos\zeta(k'-k)}e^{ik'ct}e^{ikct}, \quad (47)$$

where the integrations over k' and t in (47) are an inverse

Fourier transform of

$$\frac{T_{n',\zeta}(k')H(k')}{c} J_{n'}(k'r\sin\zeta)J_n(kr\sin\zeta)e^{iz\cos\zeta(k'-k)}, \tag{48}$$

followed by a Fourier transform, which cancel each other. With (48), (47) is simplified to (49) [page 1479]. The square brackets in (49) [page 1479] are given by (23). Substitute (23) into (49), we have:

$$\int_{-\pi}^{\pi} d\phi \int_{-\infty}^{\infty} dt\Phi(r,\phi,z-c_1t)\Phi^*_{A_{n,k,\zeta}}(r,\phi,z-c_1t) = \frac{(2\pi)^2 H(k)J_n^2(kr\sin\zeta)}{c}T_{n,\zeta}(k) \tag{50}$$

From (50), we obtain (27).

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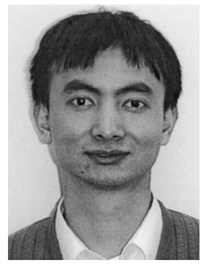
Jian-yu Lu (S'88–M'88–SM'99) was born in Fuzhou, Fujian Province, People's Republic of China. He received the B.S. degree in electrical engineering in February 1982 from Fudan University, Shanghai, China; the M.S. degree in 1985 from Tongji University, Shanghai, China; and the Ph.D. degree in 1988 from Southeast University, Nanjing, China.

He currently is a professor in the Department of Bioengineering at The University of Toledo, Toledo, OH, and an adjunct professor of medicine at the Medical College of Ohio, Toledo, OH. Before joining The University of Toledo as a professor in September 1997, he was an associate professor of biophysics at the Mayo Medical School and an associate consultant at the Department of Physiology and Biophysics, Mayo Clinic/Foundation, Rochester, MN. From March 1990 to December 1991, he was a research associate at the Department of Physiology and Biophysics; from December 1988 to February 1990, he was a postdoctoral research fellow there. Prior to that, he was a faculty member of the Department of Biomedical Engineering, Southeast University, Nanjing, China, and worked with Prof. Yu Wei.

Dr. Lu's research interests are in acoustic imaging and tissue characterization, medical ultrasonic transducers, and ultrasonic beam forming and propagation.

He is a recipient of the Outstanding Paper Award for two papers published in the 1992 *IEEE Transactions on the UFFC,* and the recipient of the Edward C. Kendall Award from the Mayo Alumni Association, Mayo Foundation in 1992. He also received both the FIRST Award from the NIH and the Biomedical Engineering Research Grant Award from the Whitaker Foundation in 1991. He is a member of the IEEE UFFC Society, the American Institute of Ultrasound in Medicine, Sigma Xi, and other societies.



Anjun Liu was born September 17, 1972, in Hubei Province, People's Republic of China. He received the B.Eng. degree in biomedical engineering in June 1995 from the Southeast University, Nanjing, China, and the M.Eng. degree in 1998 also from the Southeast University.

He currently is a predoctoral student in the Department of Bioengineering, The University of Toledo, Toledo, OH. In 1994, he practiced in the Haiying Ultrasonic Medical Equipment Factory, Nanjing, China. He also participated in a number of research and graduation projects and was a teaching assistant.

Mr. Liu won the Undergraduate Scholarship of the Southeast University (1993–1995). He received a Software Engineer Certificate issued by the National Software Technical Qualification and Level Test Center in 1996. He received a Hardware Level Certificate (Rank 3) in 1993. His hobby is sports and arts, swimming, chess, skating, and pop music.